Pergamon

Int. J. Solids Structures Vol. 35, Nos. 5–6, pp. 421–456, 1998

0020–7683/98 $19.00 + .00

PII: S0020–7683(97)00036–X

GRADIENT ELASTICITY WITH SURFACE ENERGY: MODE-I CRACK PROBLEM

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(*Received* 30 *May* 1996; *in revised form* 8 *January* 1997)

Abstract—The solution of the mode-I crack problem is given by using an anisotropic strain-gradient elasticity theory with surface energy, extending previous results by Vardoulakis and co-workers, as well as by Aifantis and co-workers. The solution of the problem is derived by applying the Fourier transform technique and the theory of dual integral and Fredholm integral equations. Asymptotic analysis of the solution close to the tip gives a cusping crack with zero slope of the crack displacement at the crack tip. Cusping of the crack tips is caused by the action of "cohesive" double forces behind and very close to the tips, that tend to bring the two opposite crack lips in close contact. Consideration of Griffith's energy balance approach leads to the formulation of a fracture criterion that predicts a linear dependence of the specific fracture surface energy on increment of crack propagation for such crack length increments that are comparable with the characteristic size of material's microstructure. This important theoretical result agrees with experimental measurements of the fracture energy dissipation rate during fracturing of polycrystalline, polyphase materials such as rocks and ceramics. The potential of the theory to interpret the size effect, i.e. the dependence of fracture toughness of the material on the size of the crack, is also presented. Also, the theory predicts an inverse first power relation between the tensile strength and the size of the pre-existing crack which is in accordance with experimental evidence. Furthermore, it is shown that the effect of the volumetric strain–gradient term is to shield the applied loads leading to crack stiffening, hence the theory captures the commonly observed phenomenon of high-effective fracture energies of rocks and ceramics; the effect of the surface strain–energy term is to amplify the applied loads leading to crack compliance and essentially captures the development of the "process zone" or microcracking zone around the main crack in a brittle material. Thus, the present anisotropic gradient elasticity theory with surface energy provides an effective tool for understanding phenomenologically main crack–microdefect interaction phenomena in brittle materials.

1. INTRODUCTION

Professor G. I. Barenblatt (1952, p. 59) in his celebrated paper stated that: "*By using the model of an elastic body, we do not take into consideration all forces acting upon the body. It appears that for developing an adequate theory of cracks it is necessary to consider molecular forces of cohesion acting near the edge of a crack, where the distance between the opposite faces of the crack is small and the mutual attraction strong.*" The first person to introduce molecular forces of cohesion acting near the tip of a crack was Griffith, who considered forces of cohesion as forces of surface tension being internal forces for the given body, in order to develop his celebrated criterion of fracture mechanics (Griffith, 1921). However, their effect on the stresses and strains was neglected by Griffith. Following a different approach, Elliot (1947) proposed an atomistic model which explicitly accounted for the effect of the interatomic forces along the crack faces, and later Barenblatt (1962) introduced a small cohesive zone ahead of the "physical" crack tip whose size is determined axiomatically by requiring the cancellation of singularity at the tip of the cohesive zone (or tip of "effective" crack).

In a recent paper (Vardoulakis *et al.*, 1996), the effect of volumetric and surface strain-gradient terms, ℓ, ℓ', respectively, which were accounted for by an anisotropic gradient elasticity theory—that implicitly accounts in a phenomenological manner for cohesive forces acting upon a body—was studied in the context of mode-III crack deformation. The aim was to further investigate the earlier questions posed by Aifantis and co-workers

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(Aifantis, 1992, 1994; Altan and Aifantis, 1992, 1996; Ru and Aifantis, 1993). In those articles the potential of applying gradient elasticity theory to lincar elastic fracture mechanics (LEFM) was explored by describing the stress and displacement fields in the vicinity of Griffith cracks. For simplicity, Aifantis and co-workers studied only the effect of the volumetric strain–gradient energy term and, as he states, his theory may be viewed in a sense, as a particular case of Mindlin's (1965) original theory, involving only one material constant. However, the concept of higher-order self-equilibrating stresses doing work on higher-order strain gradients need not be introduced. The special theory has been applied by Altan and Aifantis (1992, 1996) in order to solve the mode-I, -II and -III unclamped tips crack problems by using the Fourier transform technique, as well as by Ru and Aifantis (1993) to consider certain boundary value problems by reducing the solutions of the gradient theory to solutions of classical elasticity. Altan and Aifantis essentially hit upon the indefiniteness of the crack length, which reflects a discrete or atomic viewpoint rather than a continuum mechanics viewpoint and their results were presented in an integral form. Ru and Aifantis have found that for traction boundary value problems the stresses are identical to those of the classical theory of elasticity, whereas the crack faces close at the tip with an enclosed non-zero angle which is not realistic if we consider that molecular forces of cohesion are acting near the tip of the crack. By introducing second-order strain gradients Unger and Aifantis (1995) obtained a simple closed form asymptotic solution involving the mode-III semi-infinite clamped-tips crack, with tractionless surfaces in an infinite plate by utilizing the Westergaard stress function of the asymptotic classical elasticity solution. The analytical solution for mode-III crack eliminated the strain singularity at the crack tip and produced oscillatory profiles for the crack faces which join together in the form of a cusp. More recently, Unger and Aifantis (1996) were able to produce an exact gradient elastic–plastic solution by extending the corresponding mode-III classical small-scale yielding solution of McClintock and Rice. Furthermore, along the same line of the special gradient elasticity theory, Exadaktylos and Aifantis (1996) presented analytical and asymptotic solutions for the mixed–mixed boundary value problems of mode-I, -II, -III and penny-shaped clamped-tip cracks by applying integral transform techniques, such as Fourier and Hankel transforms, as well as the theory of dual integral and Fredholm integral equations.

Vardoulakis *et al.* (1996) deviated from this path, suggesting a constitutive model that accounts additionally for surface energy, strain–gradient terms. The main conclusions concerning the clamped-tips mode-III crack problem, were that the stresses remain the same as those predicted by classical LEFM, i.e. they are singular at the crack tip, whereas the crack tips have the form of cusps of the first kind with zero enclosed angle. The latter is consistent with Barenblatt's (1962) "cohesive-zone" theory, but without requiring the extra assumption on the existence and effect of interatomic forces, as such effects are already incorporated in the stress–strain relation of the gradient elasticity. The solution by Vardoulakis *et al.* (1996) gave finite strains at the mode-III crack tip region in contrast to LEFM which predicts strains that exhibit an inverse square root singularity at the crack tip. It was also found that the incorporation of the volumetric energy term ℓ into the governing equation leads to crack stiffening effect and, thus to high apparent fracture resistance, which is consistent with observational evidence (Friedman *et al.* 1972; Hoagland *et al.* 1973; Ortiz, 1988) concerning fracturing of brittle polycrystalline, polyphase materials such as ceramics and rocks. In these cases, there is mitigation of the applied loads by a softening of the material surrounding the crack tip (toughening or shielding mechanism). The surface energy term ℓ', which appears only in the boundary conditions, also controls the compliance of the crack and may have an opposite effect, namely a degradation or embrittlement effect. The consideration of the surface energy term leads to a constitutive character of the boundary conditions. This strengthens Aifantis' (1978) conjecture of the constitutive character of boundary constraints in materials with microstructure. Professor I. Vardoulakis, noting this fact, states (Vardoulakis *et al.* 1992, p. 581): "*The problem of constitutive boundary conditions is open and deserves further attention from the theoretical as well as the experimental point of view.*" Further on, in order to connect this theory with Barenblatt's theory of cohesive forces, Exadaktylos *et al.* (1996) considered the inverse mode-III crack problem, that is the problem of determining what distribution of pressure is necessary to produce a mode-III crack of prescribed shape. It was found that at some distance behind the crack tip Barenblatt-type "cohesion" forces are acting with the distribution of these forces to depend on the characteristic lengths ℓ, ℓ'. It was also shown, that an infinite closing stress acts at the crack tip, which changes sign at some distance from it and then takes a finite maximum value. This behavior of the stresses shows that a crack in a brittle material initiates at a finite distance in front of the crack tip and, consequently, it propagates by finite jumps. Further experimental work has to be performed in order to support this theoretical result.

Higher-order gradient theories were popular topics of research in the sixties. Mindlin's work is most noteworthy in that his goal was specifically targeted at understanding, phenomenologically, the effect of microstructure on the deformation of solids (Mindlin, 1964, 1965). However, Mindlin's (1965) isotropic grade-3, linear elasticity theory with surface energy, which was further explored as far as its mathematical potential is concerned in a comprehensive paper by Wu (1992), includes 16 material constants, plus the classical Lame's constants, whereas the present anisotropic grade-2 theory contains only the material constants ℓ, ℓ' whose determination nonetheless constitutes a formidable experimental challenge. Also a grade-2 theory is mathematically more tractable than a grade-3 theory. Mindlin's cohesive elasticity theory accounts in a phenomenological manner for molecular forces of cohesion acting upon a body, which are not considered by the classical linear elasticity theory, by including in the potential energy–density of an elastic solid, the modulus of cohesion, which is essentially an initial, homogeneous, self-equilibrating triple stress. Wu (1992) showed the apparent Young's modulus obtained from a film is higher than that obtained from a slab, thus Mindlin's theory is able to capture the scale effect of Young's modulus. Wu states in his paper (Wu, 1992, p. 102) that: "*No continuum field theories can be one hundred percent physical. For example, it could be argued that Young's modulus should be derived from more fundamental physical constants, but the 'correct' Young's modulus is always measured from a tension specimen, which was designed in accordance with the mathematical solution that* $\sigma = P/A$ *and* $\varepsilon = \Delta L/L$. *It is our belief that many, many more P/A formulae must and will be discovered in the future.*"

Casal's (1961) original idea on one-dimensional (1-D) tension bar problem, which was generalized by Vardoulakis and co-workers (Vardoulakis and Sulem, 1995; Vardoulakis *et al.*, 1996) into an anisotropic gradient elasticity theory with surface energy, predicts that the bar elongation $\Delta L/L$, in a clamped end–free end configuration with the uniaxial tension load σ acting on its free end, is given by

$$\frac{\Delta L}{L} = \frac{\sigma}{E}\left[1 + \frac{1}{\left(\frac{\ell}{\ell'}\right)^2 - 1}\frac{2\ell}{L}\,\mathrm{th}\,\frac{L}{2\ell}\right]$$

with E being the Young's modulus of the bar. That is, Young's modulus of the gradient elastic material increases as the characteristic dimension L of the body increases. In this theory the volumetric strain–gradient term ℓ expresses the scale effect exhibited by Young's modulus, whereas the surface energy term ℓ' depicts the intensity of the scale effect. Casal considered the effect of the granular, polycrystalline and atomic nature of materials on their macroscopic response through the concept of internal and superficial capillarity expressed by the material lengths ℓ, ℓ', respectively, rather than through intractable statistical mechanics concepts. The concept that the surfaces of liquids are in a state of tension is a familiar one and it is widely utilized. Actually, it is known that no skin or thin foreign surface really is in existence at the surface and that the interaction of surface molecules causes a condition analogous to a surface subjected to tension. The surface tension concept is, therefore, an analogy, but it explains the surface behavior in such a satisfactory manner that the actual molecular phenomena need not be invoked. In order to take into account the first two gradients of the displacement (grade-2 theory) he introduced, apart from the usual tension force which acts on the macroscopic strain, a self-equilibrating double tension force ("surtension") which acts on the strain–gradient. He further gives the example of the 1-D tension bar with a fixed end–fixed end configuration and showed that in this case the following relationship is valid

$$\frac{\Delta L}{L} = \frac{\sigma}{E}\left[1 - \frac{\ell}{L}\text{th}\frac{L}{2\ell}\right].$$

That is to say, in this specific bar configuration the effect of the volumetric strain–gradient term is to reduce bar elongation instead of increasing it as happens in the classical case.

Consideration is given here to the problem of the uniformly pressurized finite length mode-I crack in an infinite medium. The plan of the paper is as follows. In Section 2, the basic equations of anisotropic gradient elasticity theory with surface energy are reviewed. Section 3 contains the formulation of the half-plane mode-I crack problem in plane strain conditions with the aid of the Fourier transform technique. In Section 4, the problem is given in terms of a set of dual integral equations which are further reduced to a Fredholm integral equation of the second kind that is amenable to a numerical treatment. Section 5 gives the asymptotic solution near the crack tip, the size effect exhibited by the fracture toughness of the gradient elastic material, as well as the expression for the important physical quantity of the crack energy release rate. Graphical results are also presented in Section 6 for the variation of the crack profile and energy release rate with the volumetric and surface energy terms.

2. BASIC EQUATIONS

Higher grade continua belong to a general class of constitutive models which account for the materials microstructure. An early formulation of a simple linear continuum theory with microstructure can be found in a rather unnoticed publication by Casal (1961), referred to by Germain (1973a, b). It is noted that Casal's model cannot be directly embedded in Mindlin's (1964) linear, isotropic elasticity theory with microstructure, because the former is an anisotropic elasticity model. Instead, Casal's expression for the global strain energy of the 1-D tension bar was recovered by introducing an appropriate anisotropic, linear elastic restricted Mindlin continuum. The theory is fully presented in Vardoulakis and Sulem (1995, chapter 10), however, for easy reference we recapitulate the basic equations.

In strain–gradient dependent theory of elasticity the strain energy density function, w, is assumed to be a function of not only the first gradients, but also of the second gradients of the displacement field

$$w = w(\varepsilon_{ij}, \partial_k \varepsilon_{ij}) \tag{1}$$

where ε_{ij} is the symmetric part of the displacement field defined as follows

$$\varepsilon_{ij} = \tfrac{1}{2}(\partial_j u_i + \partial_i u_j). \tag{2}$$

In eqn (2) u_i is the Cartesian component of the displacement vector and $\partial_k \equiv \partial/\partial x_k$, with x_k to denote space coordinates. Furthermore, since we are dealing with single-valued displacement fields one can easily establish a one-to-one correspondence between $\partial_k \varepsilon_{ij}$ and $\partial_k \partial_j u_i$ (Mindlin and Eshel, 1968). Germain (1973a, b) suggested a general framework for the foundation of consistent higher grade continuum theories on the basis of the virtual work principle. This approach starts from the definition of the variation of the total potential energy in a volume V of the body with arbitrary variation of ε_{ij}. In the particular case of a restricted Mindlin continuum, i.e. a micro-homogeneous material for which the macroscopic strain coincides with the micro-deformation, this is defined as follows (Mindlin, 1964, 1965)

$$\delta \int_V w \, \mathrm{d}V = \int_V (\tau_{ij}\delta\varepsilon_{ij} + \mu_{ijk}\,\partial_i\delta\varepsilon_{jk})\,\mathrm{d}V \tag{3}$$

where

$$\tau_{ij} = \frac{\partial w}{\partial \varepsilon_{ij}}, \quad \mu_{ijk} = \frac{\partial w}{\partial(\partial_i \varepsilon_{jk})}. \tag{4}$$

The second-order stress tensor τ_{ij}, which is dual in energy to the macroscopic strain, is symmetric (i.e. $\tau_{ij} = \tau_{ji}$) and is called by Mindlin the Cauchy stress, whereas the third-order tensor μ_{ijk}, which is dual in energy to the strain–gradient is called the double stress and is symmetric with respect to the last two indices (i.e. $\mu_{ijk} = \mu_{ikj}$). To prepare for the formulation of a variational principle, we apply the chain rule of differentiation and the divergence theorem; furthermore, following Mindlin's approach (Mindlin, 1965) we resolve $\partial_i u_j$ on the boundary ∂V of V into a surface gradient and a normal gradient

$$\partial_i\delta u_j \equiv D_i\delta u_j + n_i D\delta u_j; D_i \equiv (\delta_{ik} - n_i n_k)\,\partial_k, \quad D \equiv n_k\,\partial_k \tag{5}$$

where δ_{ij} is the Kronecker delta and n_k is the outward unit normal on the boundary ∂V.

The final expression for the variation in potential energy reads

$$\delta \int_V w \, \mathrm{d}V = -\int_V \partial_i(\tau_{ij} - \mu_{kij,k})\,\delta u_j\,\mathrm{d}V + \int_{\partial V} n_i(\tau_{ij} - \mu_{ijk,k})\,\delta u_j\,\mathrm{d}S$$
$$+ \int_{\partial V} L_i n_k \mu_{ijk}\delta u_j\,\mathrm{d}S + \int_{\partial V} n_i n_k \mu_{ijk} D\delta u_j\,\mathrm{d}S \tag{6}$$

where $L_i = n_i D_k n_k - D_i$. Looking at the structure of eqn (6) we now postulate the following principle of stationary potential energy, which could also be interpreted as a principle of virtual work

$$\delta \int_V w \, \mathrm{d}V = \int_V f_j \delta u_j\,\mathrm{d}V + \int_{\partial V} (\tilde{P}_j\delta u_j + \tilde{R}_j D\delta u_j)\,\mathrm{d}S \tag{7}$$

where f_k is the body force per unit volume and $\tilde{P}_k$, $\tilde{R}_k$ are the specified tractions and double tractions, respectively, on the smooth surface ∂V. From eqns (6) and (7) the stress–equilibrium equations in the volume V is found

$$\partial_i(\tau_{ij} - \partial_k \mu_{kij}) + f_j = 0. \tag{8}$$

The surface ∂V of the considered volume V is divided into two complementary parts ∂V_u and ∂V_σ such that on ∂V_u kinematic data, whereas on ∂V_σ static data are prescribed. In classical continua these are constraints on displacements and tractions, respectively. For the stresses the following set of boundary conditions on a smooth surface ∂V_σ is also derived from the virtual work principle

$$n_j \tau_{jk} - n_i n_j D\mu_{ijk} - (n_i D_j + n_j D_i)\mu_{ijk} + (n_i n_j D_l n_l - D_j n_i)\mu_{ijk} = \tilde{P}_k \tag{9}$$

$$n_i n_j \mu_{ijk} = \tilde{R}_k. \tag{10}$$

Since second-grade models introduce second strain gradients into the constitutive description, additional kinematic data must be prescribed on ∂V_u, as is apparent from the form of eqn (7). With the displacement already give in ∂V_u, only its normal derivative with respect to that boundary is unrestricted. This means that on ∂V_u the normal derivative of the displacement should also be given, i.e.

$$u_i = w_i \quad \text{on } \partial V_{u1} \quad \text{and} \quad Du_i = r_i \quad \text{on } \partial V_{u2}. \tag{11}$$

Also, products of appropriate components of $\tilde{P}_i$ and u_i or $\tilde{R}_i$ and Du_i are apparent from eqn (7). Next, by defining the total stress tensor σ_{ij}

$$\sigma_{ij} = \tau_{ij} - \partial_k \mu_{kij} \tag{12}$$

the stress–equilibrium, eqn (8), takes the following form in the volume V

$$\partial_i \sigma_{ij} + f_j = 0 \tag{13}$$

whereas, the workless second-order relative stress tensor α_{ij} in a restricted Mindlin continuum is in equilibrium with the double stress (Mindlin, 1964)

$$\alpha_{jk} + \partial_i \mu_{ijk} = 0. \tag{14}$$

Notice that according to eqn (13) the total stress tensor is identified with the common (macroscopic) equilibrium stress tensor.

The three-dimensional (3-D) generalization of Casal's gradient-dependent anisotropic elasticity with surface energy is straightforward, leading to the following expression for the strain energy density function (Exadaktylos and Vardoulakis, 1996a)

$$\begin{aligned} w = {} & \tfrac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + G\varepsilon_{ij}\varepsilon_{ji} + \tfrac{1}{2}\lambda\ell^2\,\partial_k\varepsilon_{ii}\,\partial_k\varepsilon_{jj} + G\ell^2\,\partial_k\varepsilon_{ij}\,\partial_k\varepsilon_{ji} \\ & + \tfrac{1}{2}\lambda\ell_k\,\partial_k(\varepsilon_{ii}\varepsilon_{jj}) + G\ell_k\,\partial_k(\varepsilon_{ij}\varepsilon_{ji}) \quad i,j,k = 1,2,3 \end{aligned} \tag{15}$$

where λ and G are Lame's constants, and ℓ, ℓ' are characteristic lengths of the material defined previously and

$$\ell_k = \ell' v_k, \quad v_k v_k = 1 \tag{16}$$

is a director. Accordingly, eqn (15) defines a gradient anisotropic elasticity with constant characteristic directors ℓ_k. The last two terms in eqn (15) have the meaning of surface energy, since by using the divergence theorem

$$\int_V \ell_r\,\partial_r(\varepsilon_{pq}\varepsilon_{qp})\,\mathrm{d}V = \ell' \int_V (\varepsilon_{pq}\varepsilon_{qp})(v_r n_r)\,\mathrm{d}S. \tag{17}$$

It turns out that for positive definite strain energy density, the elastic constants λ, G and the material lengths ℓ, ℓ' are restricted, such that (Vardoulakis *et al.* 1996; Exadaktylos and Vardoulakis, 1996b)

$$3\lambda + 2G > 0, \quad G > 0, \quad -1 < \frac{\ell'}{\ell} < 1. \tag{18}$$

The first two inequalities for a bounded 3-D region are due to Kirchhoff (1859), whereas the last inequality is due to the consideration of higher displacement gradients in the strain energy density function. From eqns (4) and (15) follow the constitutive relations for the total stress, Cauchy stress and double stress tensors, respectively

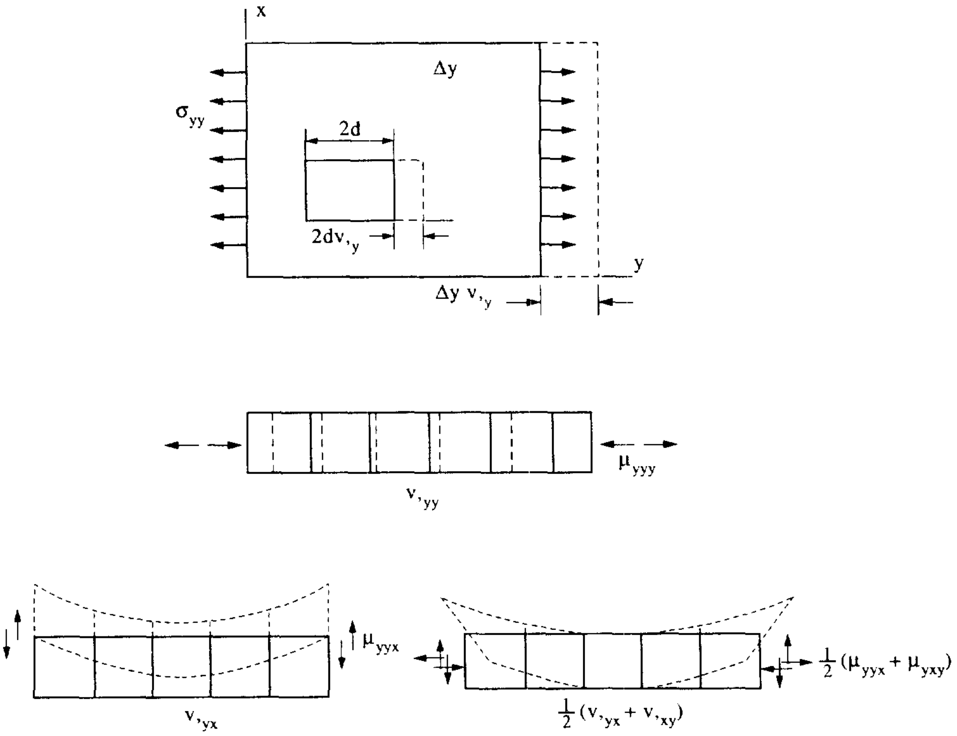

Fig. 1. Total stress σ_{yy}, displacement gradient $v,_y$ $(=\partial v/\partial y)$ and double stresses μ_{yyy}, μ_{yyx} and $\frac{1}{2}(\mu_{yyx}+\mu_{yxy})$.

$$\left.\begin{aligned} \sigma_{ij} &= \lambda\delta_{ij}\varepsilon_{kk}+2G\varepsilon_{ij}-\ell^2\nabla^2(\lambda\delta_{ij}\varepsilon_{kk}+2G\varepsilon_{ij}) \\ \tau_{ij} &= \lambda\delta_{ij}\varepsilon_{kk}+2G\varepsilon_{ij}+\ell_k\,\partial_k(\lambda\delta_{ij}\varepsilon_{kk}+2G\varepsilon_{ij}) \\ \mu_{kij} &= \ell_k(\lambda\delta_{ij}\varepsilon_{ll}+2G\varepsilon_{ij})+\ell^2\,\partial_k(\lambda\delta_{ij}\varepsilon_{ll}+2G\varepsilon_{ij}) \end{aligned}\right\} \tag{19}$$

where δ_{ij} is the Kronecker delta. It has been shown by Ru and Aifantis (1993) and by Exadaktylos and Vardoulakis (1996a), that for the case of traction boundary value problems the gradient dependent elasticity predicts the same stresses σ_{ij} with the classical theory of elasticity. The 27 components μ_{kij} have the character of double forces per unit area. The first subscript of a double stress μ_{kij} designates the normal to the surface across which the component acts; the second and third subscripts have the same significance as the two subscripts of σ_{ij} (Fig. 1). The eight components of the deviator of the couple-stress or couples per unit area formed by the combinations $1/2(\mu_{pqr}-\mu_{prq})$ are all equal to zero in the present gradient dependent elasticity theory, whereas all the remaining 10 independent combinations $1/2(\mu_{pqr}+\mu_{prq})$ are self-equilibrating (Mindlin, 1964, 1965). Double force systems without moments are stress systems equivalent to two oppositely directed forces at the same point; such systems have direction, but not net force and no resulting moment. Notice that singularities of this kind are discussed by Love (1927) and Eshelby (1951).

3. FORMULATION OF THE PROBLEM

The displacement-equation of equilibrium of the present gradient dependent elasticity theory with surface energy in the absence of body forces is (Exadaktylos and Vardoulakis, 1996a)

$$\bar{D}^2[(\lambda+G)\nabla\nabla\cdot\mathbf{u}+G\nabla^2\mathbf{u}] = 0 \tag{20}$$

where a bold symbol means that this is a vector, ∇ and $\nabla\cdot$ are the gradient and divergence operators, respectively, ∇^2 denotes the Laplacian operator and the operator $\bar{D}^2$ is defined as follows

$$\bar{D}^2 \equiv 1-\ell^2\nabla^2. \tag{21}$$

As it is observed from eqns (20) and (21), the constant ℓ', even when properly included in the constitutive equations, does not appear in the displacement equations of equilibrium. Nevertheless, it may enter the displacement field through certain of the boundary conditions.

In a Cartesian coordinate system (x, y, z) for the case of plane strain parallel to the xy-plane with

$$\mathbf{u} = (u(x,y), v(x,y), 0) \tag{22}$$

eqn (20) yields

$$\left.\begin{aligned} (\lambda+G)\bar{D}^2\,\partial_x\theta+G\bar{D}^2\nabla^2 u &= 0 \\ (\lambda+G)\bar{D}^2\,\partial_y\theta+G\bar{D}^2\nabla^2 v &= 0 \end{aligned}\right\} \tag{23}$$

where

$$\theta = \partial_x u+\partial_y v, \quad \nabla^2 = \partial_x^2+\partial_y^2.$$

The components of the infinitesimal strain tensor in plane strain are given by

$$\varepsilon_{xx} = \partial_x u, \quad \varepsilon_{yy} = \partial_y v, \quad \varepsilon_{xy} = \tfrac{1}{2}(\partial_y u+\partial_x v), \quad \varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0. \tag{24}$$

In view of the constitutive equations of anisotropic gradient elasticity with surface energy, the components of the stress tensor are found from the first of relations (19) to be

$$\left.\begin{aligned} \sigma_{xx} &= (\lambda+2G)\,\partial_x u+\lambda\,\partial_y v-\ell^2\nabla^2[(\lambda+2G)\,\partial_x u+\lambda\,\partial_y v] \\ \sigma_{yy} &= (\lambda+2G)\,\partial_y v+\lambda\,\partial_x u-\ell^2\nabla^2[(\lambda+2G)\,\partial_y v+\lambda\,\partial_x u] \\ \sigma_{xy} &= \sigma_{yx} = G(\partial_y u+\partial_x v)-\ell^2\nabla^2 G(\partial_y u+\partial_x v) \\ \sigma_{zz} &= \lambda(\partial_x u+\partial_y v)-\ell^2\nabla^2\lambda(\partial_x u+\partial_y v) \\ \sigma_{zx} &= \sigma_{zy} = 0 \end{aligned}\right\}. \tag{25}$$

The only non-vanishing components of the double stress tensor given by the third of eqn (19) for the case $\nu_k \equiv n_k$ for the half-plane $y \geqslant 0$ furnish

$$\left.\begin{aligned} \mu_{xxx} &= \ell^2\,\partial_x[(\lambda+2G)\varepsilon_{xx}+\lambda\varepsilon_{yy}] \\ \mu_{yxx} &= -\ell'[(\lambda+2G)\varepsilon_{xx}+\lambda\varepsilon_{yy}]+\ell^2\,\partial_y[(\lambda+2G)\varepsilon_{xx}+\lambda\varepsilon_{yy}] \\ \mu_{xyy} &= \ell^2\,\partial_x[(\lambda+2G)\varepsilon_{yy}+\lambda\varepsilon_{xx}] \\ \mu_{yyy} &= -\ell'[(\lambda+2G)\varepsilon_{yy}+\lambda\varepsilon_{xx}]+\ell^2\,\partial_y[(\lambda+2G)\varepsilon_{yy}+\lambda\varepsilon_{xx}] \\ \mu_{xxy} &= \mu_{xyx} = 2G\ell^2\,\partial_x\varepsilon_{xy}, \quad \mu_{yyx} = \mu_{yxy} = -2G\ell'\varepsilon_{xy}+2G\ell^2\,\partial_y\varepsilon_{xy} \end{aligned}\right\}. \tag{26}$$

Plane problems for the half-plane corresponding to eqn (13) are conveniently attacked with the aid of the Fourier transform (Sneddon, 1951). To this end we apply the following exponential Fourier transforms of the displacements

$$\left.\begin{aligned}\bar{u}(\xi,y) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,y)\,\mathrm{e}^{i\xi x}\,\mathrm{d}x\\ \bar{v}(\xi,y) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} v(x,y)\,\mathrm{e}^{i\xi x}\,\mathrm{d}x\end{aligned}\right\} \tag{27}$$

where we use the "bar notation" to denote the 1-D transform with respect to x. Also, ξ is the real-valued transform parameter and $i = \sqrt{-1}$. According to the appropriate inversion theorem (27) implies

$$\left.\begin{aligned}u(x,y) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{u}(\xi,y)\,\mathrm{e}^{-i\xi x}\,\mathrm{d}\xi\\ v(x,y) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{v}(\xi,y)\,\mathrm{e}^{-i\xi x}\,\mathrm{d}\xi\end{aligned}\right\}. \tag{28}$$

Multiplying the equations of equilibrium (13) and the stress–displacement equations (25) for plane strain by $\mathrm{e}^{i\xi x}$ and integrating along the whole x-axis, we obtain the following set of five ordinary differential equations

$$\left.\begin{aligned}&-i\xi\bar{\sigma}_{xx}+D\bar{\sigma}_{xy}=0,\quad -i\xi\bar{\sigma}_{xx}+D\bar{\sigma}_{xy}=0\\ &\bar{\sigma}_{xx} = -(i\xi)(\lambda+2G)[(1+\ell^2\xi^2)-\ell^2D^2]\bar{u}+\lambda[(1+\ell^2\xi^2)D-\ell^2D^3]\bar{v}\\ &\bar{\sigma}_{yy} = -(i\xi)\lambda[(1+\ell^2\xi^2)-\ell^2D^2]\bar{u}+(\lambda+2G)[(1+\ell^2\xi^2)D-\ell^2D^3]\bar{v}\\ &\bar{\sigma}_{xy} = G\{[(1+\ell^2\xi^2)D-\ell^2D^3]\bar{u}-(i\xi)[(1+\ell^2\xi^2)-\ell^2D^2]\bar{v}\}\end{aligned}\right\} \tag{29}$$

where $D \equiv \mathrm{d}/\mathrm{d}y$. Substituting the values of the stresses in the first two stress-equilibrium eqns (29), the following simultaneous system of coupled ordinary differential equations is obtained

$$\left.\begin{aligned}&[\ell^2D^2-(1+\ell^2\xi^2)]\{[GD^2-(\lambda+2G)\xi^2]\bar{u}-i\xi(\lambda+G)D\bar{v}\}=0\\ &[\ell^2D^2-(1+\ell^2\xi^2)\{[(\lambda+2G)D^2-G\xi^2]\bar{v}-i\xi(\lambda+G)D\bar{u}\}=0\end{aligned}\right\}. \tag{30}$$

On eliminating $\bar{u}$ and $\bar{v}$ from eqn (30) we have

$$\left.\begin{aligned}(D^2-\xi^2)^2(\ell^2D^2-[1-\ell^2\xi^2])\bar{u}=0\\ (D^2-\xi^2)^2(\ell^2D^2-[1+\ell^2\xi^2])\bar{v}=0\end{aligned}\right\}. \tag{31}$$

For the case where the gradient effects are negligible, that is $\ell \to 0$, the above conditions reduce to those of classical elasticity (Sneddon and Berry, 1958, p. 74), i.e.

$$\left.\begin{aligned}(D^2-\xi^2)^2\bar{u}(\xi,y)=0\\ (D^2-\xi^2)^2\bar{v}(\xi,y)=0\end{aligned}\right\}. \tag{32}$$

It is important to observe that in the limit $\ell \to 0$ the highest derivative term in eqn (31) is lost, suggesting the emergence of boundary layer effects. The general solution for the half-plane $y \geqslant 0$, considering that the displacements must remain finite as $y \to \infty$, as well as being symmetric with respect to the y-axis, furnishes

$$\left.\begin{aligned}\bar{u}(\xi,y) &= \{A_1(\xi)+yB_1(\xi)\}\,\mathrm{e}^{-y|\xi|}+C_1(\xi)\,\mathrm{e}^{-y\sqrt{\xi^2+(1/\ell^2)}}\\ \bar{v}(\xi,y) &= \{A_2(\xi)+yB_2(\xi)\}\,\mathrm{e}^{-y|\xi|}+C_2(\xi)\,\mathrm{e}^{-y\sqrt{\xi^2+(1/\ell^2)}},\quad y\geqslant 0,\quad -\infty<\xi<\infty\end{aligned}\right\} \tag{33}$$

where $A_i(\xi)$, $B_i(\xi)$, $C_i(\xi)$, $(i = 1,2)$ are unknown complex functions to be determined from the boundary conditions of the problem. The components of stress and double stress in the transform domain can be expressed in terms of the above unknown functions. Herein we record the stress components

$$\left.\begin{aligned}\bar{\sigma}_{xx}(\xi,y)/G &= (-i\xi)(m+2)[A_i(\xi)+(y+2\ell^2|\xi|)B_i(\xi)\}\,\mathrm{e}^{iy|\xi|}\\ &\quad +m[-|\xi|A_2(\xi)+(1-y|\xi|-2\ell^2\xi^2)B_2(\xi)]\,\mathrm{e}^{-y|\xi|}\\ \bar{\sigma}_{yy}(\xi,y)/G &= (-i\xi)m[A_1(\xi)+(y+2\ell^2|\xi|)B_1(\xi)]\,\mathrm{e}^{-y|\xi|}\\ &\quad +(m+2)[-|\xi|A_2(\xi)+(1-y|\xi|-2\ell^2\xi^2)B_2(\xi)]\,\mathrm{e}^{-y|\xi|}\\ \bar{\sigma}_{xy}(\xi,y)/G &= \bar{\sigma}_{xy}(\xi,y)/G = [-|\xi|A_1(\xi)+(1-|\xi|y-2\ell^2\xi^2)B_1(\xi)]\,\mathrm{e}^{-y|\xi|}\\ &\quad -(i\xi)[A_2(\xi)+(y+2\ell^2|\xi|)B_2(\xi)]\,\mathrm{e}^{-y|\xi|},\quad y\geqslant 0,\quad -\infty<\xi<\infty\end{aligned}\right\} \tag{34}$$

and only the following double stress components

$$\left.\begin{aligned}\bar{\mu}_{yyy}(\xi,y)/G &= (i\xi)m\{(\ell'+\ell^2|\xi|)A_1(\xi)\,\mathrm{e}^{-y|\xi|}+[\ell'y-\ell^2(1-y|\xi|)]B_1(\xi)\,\mathrm{e}^{-y|\xi|}\\ &\quad +(\ell'+\ell^2a(\xi))C_1(\xi)\,\mathrm{e}^{-ya(\xi)}\}+(m+2)\{(\ell'|\xi|+\ell^2\xi^2)A_2(\xi)\,\mathrm{e}^{-y|\xi|}\\ &\quad +[-\ell'(1-y|\xi|)+\ell^2(-2|\xi|+y\xi^2)]B_2(\xi)\,\mathrm{e}^{-y|\xi|}\\ &\quad +a(\xi)(\ell'+\ell^2a(\xi))C_2(\xi)\,\mathrm{e}^{-ya(\xi)}\}\\ \bar{\mu}_{yyx}(\xi,y)/G &= (\ell'|\xi|+\ell^2\xi^2)A_1(\xi)\,\mathrm{e}^{-y|\xi|}\\ &\quad +[-\ell'(1-y|\xi|)+\ell^2(-2|\xi|+y\xi^2)]B_1(\xi)\,\mathrm{e}^{-y|\xi|}+a(\xi)(\ell'+\ell^2a(\xi))C_1(\xi)\,\mathrm{e}^{-ya(\xi)}\\ &\quad +(i\xi)\{(\ell'+\ell^2|\xi|)A_2(\xi)\,\mathrm{e}^{-y|\xi|}+[\ell'y-\ell^2(1-y|\xi|)]B_2(\xi)\,\mathrm{e}^{-y|\xi|}\\ &\quad +(\ell'+\ell^2a(\xi))C_2(\xi)\,\mathrm{e}^{-ya(\xi)}\},\quad y\geqslant 0,\quad -\infty<\xi<\infty\end{aligned}\right\} \tag{35}$$

where we have put $m = \lambda/G = 2v/(1-2v)$ and

$$a(\xi) \equiv \sqrt{\xi^2+\frac{1}{\ell^2}}. \tag{36}$$

We now state the mixed plane strain boundary value problem of a finite straight mode-I crack (Griffith crack) occupying the line segment $-\alpha < x < \alpha$, $y = \pm 0$ subject to a uniform internal pressure $-\sigma_0$, with σ_0 being a constant positive number, with no loading at infinity (Sternberg and Muki, 1967). Let S be the complement of the line segment $-\alpha < x < \alpha$, $y = 0$ extended on the half-plane $y \geqslant 0$. We seek the solution in S subject to the following mixed boundary conditions (derived from the virtual work principle)

$$\left.\begin{aligned}\sigma_{yy} &= -\sigma_0\\ \mu_{yyy} &= 0\qquad 0\leqslant x<\alpha,\quad y=0\end{aligned}\right\} \tag{37}$$

$$v = 0\quad \alpha<x<\infty,\quad y=0 \tag{38}$$

$$\left.\begin{aligned} &\sigma_{xy}=0\\ &\mu_{yyx}=0 \quad 0\leqslant x<\infty,\quad y=0 \end{aligned}\right\} \tag{39}$$

as well as to the homogeneous regularity conditions at infinity

$$\sigma_{ij}\to 0(i,j=x,y),\quad \mu_{ijk}\to 0(i,j,k=x,y)\quad \text{as}\sqrt{x^2+y^2}\to\infty. \tag{40}$$

The first conditions in eqns (37)–(39) are the classical ones, whereas the remaining conditions are extra boundary conditions required as a result of higher-order terms in the constitutive equations. The symmetry with respect to the y-axis (x,z-plane) provides additional conditions

$$\left.\begin{aligned} &v(x,0)=v(-x,0)\\ &\sigma_{yy}(x,0)=\sigma_{yy}(-x,0)\\ &\mu_{yyy}(x,0)=\mu_{yyy}(-x,0)\quad -\infty<x<\infty \end{aligned}\right\}. \tag{41}$$

4. REDUCTION OF THE PROBLEM TO AN INTEGRAL EQUATION

The method of solution consists of expressing the unknown functions $A_1(\xi)$, $A_2(\xi)$, $B_1(\xi)$, $C_1(\xi)$, $C_2(\xi)$ in terms of $B_2(\xi)$. This is done on the basis of eqns (30) and (33), as well as on certain of the conditions (37)–(39), while the remaining boundary conditions are reduced into a system of dual integral equations for $B_2(\xi)$ only.

We first note that the expressions for the displacements (33) must satisfy the original eqns (30), implying that the following relationships hold true

$$B_1(\xi)=iB_2(\xi),\quad A_1(\xi)=i\left[A_2(\xi)-\frac{(m+3)}{\xi(m+1)}B_2(\xi)\right]\quad 0\leqslant\xi<\infty. \tag{42}$$

Also, it is valid that

$$\sigma_{xy}|_{y=0}=0\Leftrightarrow\bar{\sigma}_{xy}|_{y=0}=0. \tag{43}$$

The shear stress $\bar{\sigma}_{xy}$ at $y=0$ can be found from eqn $(34)_3$ to be

$$\bar{\sigma}_{xy}|_{y=0}=G\{\xi A_2(\xi)+2\ell^2\xi^2B_2(\xi)+i[-\xi A_1(\xi)+(1-2\ell^2\xi^2)B_1(\xi)]\}\quad 0\leqslant\xi<\infty. \tag{44}$$

From conditions (43) and (44) and from $(42)_1$ we find

$$4\ell^2\xi^2B_2(\xi)=i\xi A_1(\xi)-\xi A_2(\xi)+B_2(\xi)\quad 0\leqslant\xi<\infty. \tag{45}$$

Combining $(42)_2$ and the previous relation (45) we get

$$A_2(\xi)=\frac{1}{\xi}\left(\frac{m+2}{m+1}-2\ell^2\xi^2\right)B_2(\xi)\quad 0\leqslant\xi<\infty. \tag{46}$$

Similarly, from the second of relations (42) and relation (46) we find

$$A_1(\xi)=-\frac{i}{\xi}\left(\frac{1}{m+1}+2\ell^2\xi^2\right)B_2(\xi)\quad 0\leqslant\xi<\infty. \tag{47}$$

In view of the relations $(34)_2$, $(42)_1$, (46) and (47)

$$\bar{\sigma}_{yy}(\xi,0)/G = -2B_2(\xi) \quad 0 \leqslant \xi < \infty. \tag{48}$$

From the first boundary conditions (37), as well as from condition $(41)_2$, we get

$$F_c[B_2(\xi);\xi \to x] = \frac{\sigma_0}{2G} \quad 0 \leqslant x < \alpha \tag{49}$$

where we have used the notation

$$F_c[\phi(\xi);\xi \to x] = \sqrt{\frac{2}{\pi}} \int_0^\infty \phi(\xi)\cos x\xi \, d\xi. \tag{50}$$

The solution of eqn (49) is (Vardoulakis *et al.*, 1996)

$$B_2(\xi) = \sqrt{\frac{\pi}{2}} \frac{\sigma_0 \alpha}{2G} J_1(\alpha\xi) \quad 0 \leqslant \xi < \infty \tag{51}$$

where $J_n(\cdot)$ is the usual Bessel function of the first kind and of order n. It turns out that

$$\sigma_{xx}(x,0) = \sigma_{yy}(x,0) = -\sigma_0\alpha \int_0^\infty J_1(\alpha\xi)\cos x\xi \, d\xi \quad 0 \leqslant x < \infty. \tag{52}$$

The integral representation for the crack displacement along the y-direction is given by

$$v(x,0) = \sqrt{\frac{2}{\pi}} \int_0^\infty [A_2(\xi) + C_2(\xi)]\cos x\xi \, d\xi \quad 0 \leqslant x < \infty. \tag{53}$$

By substituting into the above equation the expression for $A_2(\xi)$ as given by eqn (46), we obtain

$$v(x,0) = \sqrt{\frac{2}{\pi}} \frac{(m+2)}{(m+1)} \int_0^\infty \frac{1}{\xi} B_2(\xi)\cos x\xi \, d\xi + \sqrt{\frac{2}{\pi}} \int_0^\infty [-2\ell^2\xi B_2(\xi) + C_2(\xi)]\cos x\xi \, d\xi \quad 0 \leqslant x < \infty. \tag{54}$$

Furthermore, substituting the value for B_2 as given by eqn (51) into the first integral of eqn (54), we get

$$v(x,0) = \begin{cases} \frac{(m+2)}{2(m+1)} \frac{\sigma_0}{G} \sqrt{\alpha^2 - x^2} + \sqrt{\frac{2}{\pi}} \int_0^\infty [-2\ell^2\xi B_2(\xi) + C_2(\xi)]\cos x\xi \, d\xi & 0 \leqslant x < \alpha \\ \sqrt{\frac{2}{\pi}} \int_0^\infty [-2\ell^2\xi B_2(\xi) + C_2(\xi)]\cos x\xi \, d\xi & \alpha < x < \infty \end{cases}. \tag{55}$$

The crack displacement is composed of two parts, one part which is the familiar LEFM solution (Sneddon and Lowengrub, 1969, p. 29), and another which takes into account strain–gradient effects.

Our task is now to explicitly determine the function $C_2(\xi)$. To this end note that the boundary condition $(29)_2$ can be written in the following form

$$\mu_{yyx}|_{y=0} = 0 \Leftrightarrow \bar{\mu}_{yyx}|_{y=0} = 0 \tag{56}$$

which in accordance to eqns $(35)_2$, $(42)_1$, (46) and (47) yields

$$C_1(\xi) = \frac{i\xi}{a(\xi)(\ell'+\ell^2 a(\xi))}[2\ell^2(2\ell'\xi+2\ell^2\xi^2+1)B_2(\xi)-(\ell'+\ell^2 a(\xi))C_2(\xi)] \quad 0 \leqslant \xi < \infty. \tag{57}$$

Furthermore, in view of eqns $(35)_1$ and $(55)_2$, the satisfaction of the second of conditions (37) and the first of eqn (38) leads to the following dual integral equations

$$\left.\begin{aligned}
&\int_0^\infty \{m(i\xi)[(\ell'+\ell^2\xi)A_1(\xi)-\ell^2 B_1(\xi)+(\ell'+\ell^2 a(\xi))C_1(\xi)]+(m+2)[\xi(\ell'+\ell^2\xi) \\
&\quad \times A_2(\xi)-(\ell'+2\ell^2\xi^2)B_2(\xi)+a(\xi)(\ell'+\ell^2 a(\xi))C_2(\xi)]\}\cos x\xi\,\mathrm{d}\xi = 0 \quad 0 \leqslant x < \alpha \\
&\int_0^\infty \{-2\ell^2\xi B_2(\xi)+C_2(\xi)\}\cos x\xi\,\mathrm{d}\xi = 0 \quad \alpha < x < \infty
\end{aligned}\right\}. \tag{58}$$

Substituting in eqn $(58)_1$ the values for the functions A_1, B_1 and C_1, as they are given by eqns (47), $(42)_1$ and (57), respectively, we deduce

$$\left.\begin{aligned}
&\int_0^\infty \left\{\frac{2}{a(\xi)}[(\ell'+\ell a(\xi)-m\ell^2\xi^2)-2\ell^2\xi^2(m\xi-a(\xi))(\ell'+\ell^2\xi)]B_2(\xi)\right. \\
&\quad \left.+\frac{1}{a(\xi)}(\ell'+\ell^2 a(\xi))[m\xi^2+(m+2)a^2(\xi)]C_2(\xi)\right\}\cos x\xi\,\mathrm{d}\xi = 0 \qquad 0 \leqslant x < \alpha \\
&\int_0^\infty \{-2\ell^2\xi B_2(\xi)+C_2(\xi)\}\cos x\xi\,\mathrm{d}\xi = 0 \qquad \alpha < x < \infty
\end{aligned}\right\}. \tag{59}$$

Next, we assume the following Riemann–Liouville fractional integral representation for the left-hand side integral $(59)_2$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \{-2\ell^2\xi B_2(\xi)+C_2(\xi)\}\cos x\xi\,\mathrm{d}\xi = \int_x^\alpha \psi(t)\sqrt{t^2-x^2}\,\mathrm{d}t \tag{60}$$

where ψ is an integrable, sectionally continuous function that may exhibit weak singularities in the closed interval $[0, \alpha]$ and is allowed to depend on ℓ, ℓ', α and ν. By using the above relation (60), it can be easily shown that the displacement $v(x, 0)$ vanishes outside the crack region, thus the second of eqn (59) is identically satisfied.

Furthermore, from eqn (60) and the inversion theorem for the Fourier cosine transform, it follows that

$$C_2(\xi) = 2\ell^2\xi B_2(\xi) + \sqrt{\frac{\pi}{2}}\xi^{-1}\int_0^{\alpha} t\psi(t)J_1(t\xi)\,\mathrm{d}t \quad 0 \leqslant \xi < \infty. \tag{61}$$

Hence, function ψ may hereafter be regarded as the basic unknown. Next, the expressions (61) and (51) for C_2 and B_2 are introduced into the first of eqn (59), x is replaced by x', and an integration with respect to x', over the range $0 \leqslant x' \leqslant x$ $(0 \leqslant x < \alpha)$ is performed to yield

$$\int_0^{\alpha} t\psi(t)\,\mathrm{d}t\int_0^{\infty}\frac{1}{\ell^2\xi^2}\left[\frac{m\ell'\xi^2}{a(\xi)} + a(\xi)(m+2)(\ell' + \ell^2 a(\xi)) + m\ell^2\xi^2\right]J_1(t\xi)\sin x\xi\,\mathrm{d}\xi$$

$$= -\frac{\sigma_0\alpha}{G\ell^2}\int_0^{\infty}\frac{1}{\xi a(\xi)}\{(a(\xi)-\xi)[2m\ell^4\xi^3 + m\ell^2\xi - 2\ell'\ell^2\xi^2]$$

$$+ a(\xi)(\ell' + 2\ell^2\xi) + (m+2)\ell'\xi\}J_1(t\xi)\sin x\xi\,\mathrm{d}\xi \quad 0 \leqslant x < \alpha. \tag{62}$$

Equation (62) is a linear Fredholm integral equation of the first kind for the unknown function $\psi(t)$. By introducing the non-dimensional variables ζ, χ, ρ, ω, as well as, the length-ratio k defined by

$$\zeta = \ell\xi, \quad \chi = x/\ell, \quad \rho = t/\ell, \quad \omega = \alpha/\ell, \quad k = \ell'/\ell \tag{63}$$

eqn (62) furnishes

$$\int_0^{\alpha}\rho\psi(\ell\rho)K(\chi,\rho)\,\mathrm{d}\rho = F(\chi) \quad 0 \leqslant \chi < \omega \tag{64}$$

with the kernel $K(\chi, \rho)$ given by

$$\left.\begin{aligned} K(\chi,\rho) &= K_1(\chi,\rho) + K_2(\chi,\rho) \\ K_1(\chi,\rho) &= 2(m+1)\int_0^{\infty} J_1(\rho\zeta)\sin\chi\zeta\,\mathrm{d}\zeta \\ K_2(\chi,\rho) &= \int_0^{\infty} b(\zeta)J_1(\rho\zeta)\sin\chi\zeta\,\mathrm{d}\zeta \\ b(\zeta) &= (m+2)\zeta^{-2}(1+k\sqrt{1+\zeta^2}) + \frac{km}{\sqrt{1+\zeta^2}} \end{aligned}\right\} \tag{65}$$

and the free term $F(\chi)$ given by

$$F(\chi) = -\frac{\sigma_0\omega\ell^{-1}}{2G}\int_0^{\infty}\left\{2(m+2) + \frac{2k}{\zeta} + \frac{2(m+2)k}{\sqrt{1+\zeta^2}} + 4\zeta(m\zeta - k)\right.$$

$$\left. - \frac{\zeta}{\sqrt{1+\zeta^2}}(4m\zeta^2 + 2m - 4k\zeta)\right\}J_1(\omega\zeta)\sin\chi\zeta\,\mathrm{d}\zeta \quad 0 \leqslant \chi < \omega. \tag{66}$$

Taking into consideration the value of the discontinuous integral (Gradshteyn and Ryzhik, 1980)

$$K_1(\chi,\rho) = 2(m+1)\frac{\chi}{\rho}\frac{1}{\sqrt{\rho^2-\chi^2}} \quad \rho > \chi \tag{67}$$

integral eqn (64) takes the form of the Abel integral equation

$$\int_\chi^\omega \frac{\psi(\ell\rho)}{\sqrt{\rho^2-\chi^2}}\,\mathrm{d}\rho = \frac{1}{2(m+1)}f(\chi) \quad 0 \leqslant \chi < \omega \tag{68}$$

provided

$$f(\chi) = \left\{\frac{F(\chi)}{\chi} - \int_0^\omega \tau\psi(\tau)\,\mathrm{d}\tau \int_0^\infty \zeta\gamma(\chi\zeta)b(\zeta)J_1(\tau\zeta)\,\mathrm{d}\zeta\right\} \quad 0 \leqslant \chi < \omega \tag{69}$$

and

$$\gamma(\zeta) = \frac{\sin\zeta}{\zeta}. \tag{70}$$

The singular integral eqn (68) has the solution (Sneddon, 1966)

$$\psi(\ell\rho) = -\frac{1}{2(m+1)}\left[\frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^\omega \frac{\chi f(\chi)\,\mathrm{d}\chi}{\sqrt{\chi^2-\rho^2}}\right] \quad 0 \leqslant \rho < \omega. \tag{71}$$

Since $f(\chi)$ is differentiable, integration by parts and subsequent differentiation leads to

$$\psi(\ell\rho) = -\frac{1}{2(m+1)}\frac{2\rho}{\pi}\int_\rho^\omega \frac{1}{\sqrt{\chi^2-\rho^2}}\frac{\mathrm{d}}{\mathrm{d}\chi}[f(\chi)]\,\mathrm{d}\chi \quad 0 \leqslant \rho < \omega. \tag{72}$$

Following the same procedure as in Vardoulakis *et al.* (1996) we finally obtain (for easy reference see also Appendix A)

$$\begin{aligned}\phi(\rho) = &-\frac{1}{2(m+1)}\int_0^\omega \tau\phi(\tau)\,\mathrm{d}\tau\int_0^\infty \zeta b(\zeta)J_1(\rho\zeta)J_1(\tau\zeta)\,\mathrm{d}\zeta \\ &-\frac{1}{4(m+1)}\frac{\sigma_0\omega}{G}\int_0^\infty c(\zeta)J_1(\rho\zeta)J_1(\omega\zeta)\,\mathrm{d}\zeta - \frac{1}{4(m+1)}\frac{\sigma_0}{G}\frac{k\rho}{\omega}\,{}_2F_1\left(\frac{3}{2},\frac{1}{2};2;\frac{\rho^2}{\omega^2}\right) \\ &-\frac{(m+2)}{2(m+1)}\frac{\sigma_0}{G}\frac{\omega^{1/2}}{\rho^{1/2}}\delta(\omega-\rho) \quad 0 \leqslant \rho < \omega\end{aligned} \tag{73}$$

where we have set

$$\phi(\rho) = \ell\psi(\ell\rho), c(\zeta) = \frac{2(m+2)k\zeta - 2m\zeta^2 + 4\zeta^2(m\zeta-k)(\sqrt{1+\zeta^2}-\zeta)}{\sqrt{1+\zeta^2}} \tag{74}$$

$\delta(\cdot)$ is the generalized delta function of Dirac, ${}_2F_1(a,b;c;z)$ is Gauss's hypergeometric function. Since in our case $c-a-b = 0$, the series representation of the hypergeometric function

$$ {}_2F_1(a,b;c;z) = 1+\frac{ab}{1!c}z+\frac{a(a+1)b(b+1)}{2!c(c+1)}z^2+,\cdots, $$

converges absolutely throughout [0, 1), whereas it exhibits a logarithmic singularity in the limit as $z \to 1$

$$ {}_2F_1(a,b;c;z)|_{z\to 1} = -\frac{2}{\pi z^2}[\log(1-z^2)+\mathrm{O}(1)] \quad 0<z<1 $$

where, throughout this paper, the order-of-magnitude symbols "O" and "o" are used in their standard mathematical connotation (Erdelyi, 1956); in particular, a function is O(1) if it remains bounded in the underlying limit, whereas it is o(1) if it vanishes in the underlying limit.

The next step is to simplify the integral eqn (73) by evaluating the second integral in the right-hand part of eqn (73). First, we split this integral into two parts, one independent of the material length–ratio k, and the other dependent on the same ratio, as follows

$$ \left.\begin{aligned} I(\rho;\omega,k) &= I_\alpha(\rho;\omega)+I_\beta(\rho;\omega,k) \\ I_a(\rho;\omega) &= \int_0^\infty \frac{-2m\zeta^2+4m\zeta^3(\sqrt{1+\zeta^2}-\zeta)}{\sqrt{1+\zeta^2}}J_1(\rho\zeta)J_1(\omega\zeta)\,\mathrm{d}\zeta \\ I_b(\rho;\omega,k) &= k\int_0^\infty \frac{2(m+2)\zeta-4\zeta^2(\sqrt{1+\zeta^2}-\zeta)}{\sqrt{1+\zeta^2}}J_1(\rho\zeta)J_1(\omega\zeta)\,\mathrm{d}\zeta \end{aligned}\right\}. \tag{75} $$

Whereas, we have so far, for the sake of brevity, suppressed the arguments ℓ, k of various functions that depend on these parameters, it is helpful for our present purpose to make their ℓ- or k-dependence explicitly apparent. Accordingly, if $f(\chi)$ are values of a further function that depends, say on k as well, we now write $f(\chi;k)$ in place of $f(\chi)$. Further we set

$$ f_a(\zeta) = 2m\zeta^2\frac{-1+2\zeta(\sqrt{1+\zeta^2}-\zeta)}{\sqrt{1+\zeta^2}} \cong -0.5m\frac{\zeta}{1+\zeta^2}. \tag{76} $$

The expansions of the original function f_a and its approximation f_a^{appr} at infinity are the same and are given by

$$ f_a = f_a^{\mathrm{appr}} = -\tfrac{1}{2}\zeta^{-1}+\mathrm{o}(\zeta^{-3}) \quad \text{as } \zeta\to\infty. $$

Closely similar ideas associated with the construction of approximate solutions by approximating kernels have also been developed by Koiter (1954). Similarly, the kernel appearing in the integral I_b can be approximated by

$$ f_b(\zeta) = \frac{2(m+2)\zeta-4\zeta^2(\sqrt{1+\zeta^2}-\zeta)}{\sqrt{1+\zeta^2}} \cong 2(m+1)-2\,\mathrm{e}^{-1.9\zeta}-1.9m\,\mathrm{e}^{-\zeta} \tag{77} $$

with

$$ f_b = f_b^{\mathrm{appr}} = 2(m+1)-\mathrm{o}(\zeta^{-2}) \quad \text{as } \zeta\to\infty. $$

In view of the formulae

$$\left.\begin{aligned}
&\int_0^\infty J_1(\rho\zeta)J_1(\omega\zeta)\,d\zeta = \frac{\rho}{2\omega^2}\,{}_2F_1\left(\frac{3}{2},\frac{1}{2};2;\left[\frac{\rho}{\omega}\right]^2\right)\\
&\int_0^\infty e^{-d\zeta}J_1(\rho\zeta)J_1(\omega\zeta)\,d\zeta = \frac{1}{\pi\sqrt{\rho\omega}}Q_{1/2}\left(\frac{\omega^2+\rho^2+d^2}{2\omega\rho}\right)\\
&\int_0^\infty \frac{\zeta}{1+\zeta^2}J_1(\rho\zeta)J_1(\omega\zeta)\,d\zeta = I_1(\rho)K_1(\omega)
\end{aligned}\right\} \tag{78}$$

and the approximate expressions (76) and (77), the integrals I_a and I_b defined in eqn (75) take the form

$$\left.\begin{aligned}
&I_a(\rho;\omega) \cong -\frac{m}{2}I_1(\rho)K_1(\omega)\\
&I_b(\rho;\omega,k) \cong k\left\{(m+1)\frac{\rho}{\omega^2}\,{}_2F_1\left(\frac{3}{2},\frac{1}{2};2;\frac{\rho^2}{\omega^2}\right)\right.\\
&\left.-\frac{2}{\pi\sqrt{\rho\omega}}Q_{1/2}\left(\frac{\rho^2+\omega^2+3.6}{2\rho\omega}\right)-\frac{1.9m}{\pi\sqrt{\rho\omega}}Q_{1/2}\left(\frac{\rho^2\omega^2+1}{2\rho\omega}\right)\right\}
\end{aligned}\right\} \tag{79}$$

where $Q_{1/2}(\cdot)$ is the Legendre function of the second kind and $I_1(\cdot)$, $K_1(\cdot)$ are the modified Bessel functions of the first-order. Alternative representations of the integrals I_a and I_b can be readily deduced by means of suitable contour integrations, the details of which are omitted here. However, the complex integration technique is outlined in Appendix B concerning two other improper integral representations. The results obtained in this manner have the following form

$$\left.\begin{aligned}
&I_a = \frac{4m}{\pi}\int_0^1 Y^2\frac{1-2Y^2}{\sqrt{1-Y^2}}I_1(\rho Y)K_1(\omega Y)\,dY\\
&I_b = \frac{8k}{\pi}\int_0^1 Y^2 I_1(\rho Y)K_1(\omega Y)\,dY\\
&+\frac{4k}{\pi}\int_0^1\frac{2\sqrt{1-Y^2}+(m+2)Y^2-2}{Y^4\sqrt{1-Y^2}}I_1(\rho/Y)K_1(\omega/Y)\,dY
\end{aligned}\right\}. \tag{80}$$

The comparison of the contour integration results (80) with the approximations given by eqn (79) for the two integral representations, are presented graphically in Figs 2 and 3, respectively. In the case of the integral I_a the relative error of approximation does not exceed 4% for $0 \leqslant \rho/\omega \leqslant 1$, whereas in the case of I_b the relative error of approximation does not exceed 3% for $0 \leqslant \rho/\omega < 1$.

By virtue of approximations (76), (77) and relationships (74) the integral eqn (73) takes the form

$$\begin{aligned}
\phi(\rho) = &-\frac{1}{2(m+1)}\int_0^\omega \tau\phi(\tau)\,d\tau\int_0^\infty \zeta b(\zeta;k)J_1(\rho\zeta)J_1(\tau\zeta)\,d\zeta\\
&-\frac{(m+2)}{4(m+1)}\frac{\sigma_0}{G}\frac{k\rho}{\omega}\,{}_2F_1\left(\frac{3}{2},\frac{1}{2};2;\frac{\rho^2}{\omega^2}\right)\\
&+\frac{1}{4(m+1)}\frac{\sigma_0}{G}\frac{k}{\pi}\sqrt{\frac{\omega}{\rho}}\left[2Q_{1/2}\left(\frac{\rho^2+\omega^2+3.6}{2\rho\omega}\right)\right.
\end{aligned}$$

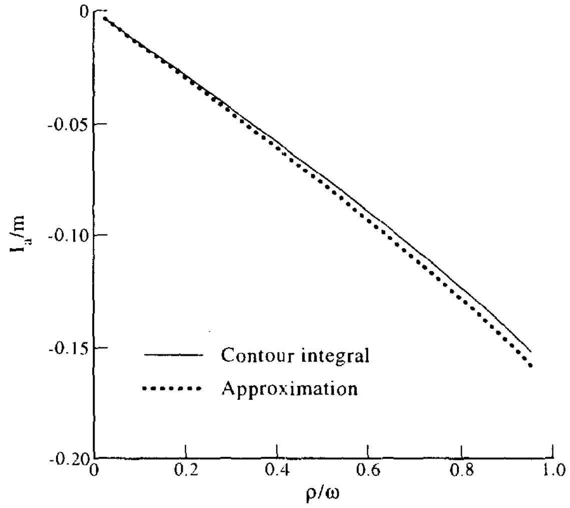

Fig. 2. The original normalized integral I_a derived by contour integration and given by eqn $(80)_1$ and its approximation $(79)_1$; $m = 2\nu/(1-2\nu)$.

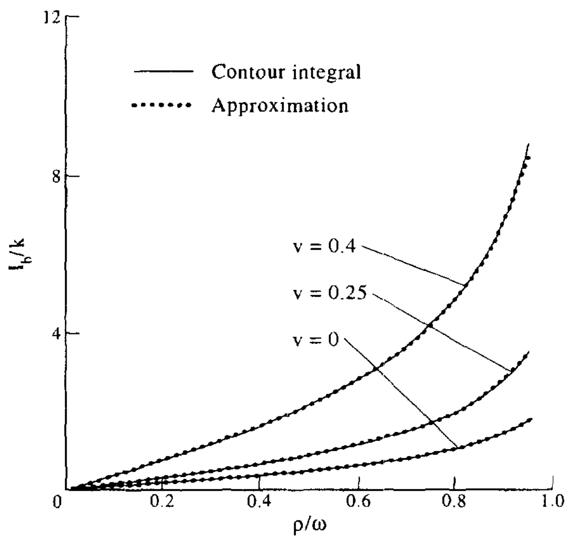

Fig. 3. The original normalized integral I_b computed by contour integration $(80)_2$ and its approximation $(79)_2$ for three values of the Poisson's ratio ν ($k = \ell'/\ell$).

$$
+1.9mQ_{1/2}\left(\frac{\rho^2+\omega^2-1}{2\rho\omega}\right)\Bigg]+\frac{0.5m}{4(m+1)}\frac{\sigma_0}{G}\omega I_1(\rho)K_1(\omega)
$$

$$
-\frac{(m+2)}{2(m+1)}\frac{\sigma_0}{G}\frac{\omega^{1/2}}{\rho^{1/2}}\delta(\omega-\rho) \quad 0 \leqslant \rho < \omega. \tag{81}
$$

The kernel of eqn (81) may be symmetrized by introducing the function

$$
\Phi(\rho) = \sqrt{\rho}\phi(\rho). \tag{82}
$$

Indeed, eqn (81) now assumes the form

$$
\Phi(\rho) - \lambda\int_0^{\omega} K^*(\rho,\tau)\Phi(\tau)\,\mathrm{d}\tau = R(\rho) \quad 0 \leqslant \rho < \omega \tag{83}
$$

where

$$\lambda = -\frac{1}{2(m+1)} \tag{84}$$

$$\begin{aligned} R(\rho) = \frac{\sigma_0}{G}\Bigg\{ & -\frac{(m+2)}{2(m+1)}\sqrt{\omega}\delta(\omega-\rho) + \frac{m}{8(m+1)}\sqrt{\rho\omega}I_1(\rho)K_1(\omega) \\ & +k\Bigg[-\frac{(m+2)}{4(m+1)}\frac{\rho\sqrt{\rho}}{\omega}\,{}_2F_1\left(\frac{3}{2},\frac{1}{2};2;\frac{\rho^2}{\omega^2}\right) \\ & +\frac{1}{4(m+1)\pi}\sqrt{\omega}\Bigg[2Q_{1/2}\left(\frac{\rho^2+\omega^2+3.6}{2\rho\omega}\right. \\ & \left.+1.9mQ_{1/2}\left(\frac{\rho^2+\omega^2+1}{2\rho\omega}\right)\right)\Bigg]\Bigg]\Bigg\} \quad 0 \leqslant \rho < \omega \end{aligned} \tag{85}$$

and

$$\begin{aligned} & K^*(\rho,\tau) = \sqrt{\rho\tau}\int_0^\infty \zeta b(\zeta;k)J_1(\rho\zeta)J_1(\tau\zeta)\,\mathrm{d}\zeta \quad 0 < \rho \leqslant \omega, 0 < \tau \leqslant \omega \\ & K^*(\rho,0) = K^*(0,\rho) = 0 \quad 0 \leqslant \rho \leqslant \omega. \end{aligned} \tag{86}$$

Equation (83) is a regular integral equation of Fredholm's second kind with a symmetric kernel. The free term, $R(\rho)$, of the integral equation consists of a generalized delta function of Dirac and a function continuous on $0 \leqslant \rho < \omega$ (the hypergeometric function exhibits a logarithmic singularity in the limit as $\rho/\omega \to 1$). The symmetric kernel K^* defined by eqn (86) can be further decomposed as follows

$$K^*(\rho,\tau) = K_1^*(\rho,\tau)+kK_2^*(\rho,\tau) \tag{87}$$

with

$$K_1^*(\rho,\tau) = (m+2)\sqrt{\rho\tau}\int_0^\infty \zeta^{-1}J_1(\rho\zeta)J_1(\tau\zeta)\,\mathrm{d}\zeta = \begin{cases} \dfrac{(m+2)}{2}\sqrt{\rho\tau}\dfrac{\tau}{\rho} & \tau \leqslant \rho \\ \dfrac{(m+2)}{2}\sqrt{\rho\tau}\dfrac{\rho}{\tau}, & t \geqslant \rho \end{cases} \tag{88}$$

and

$$K_2^*(\rho,\tau) = \sqrt{\rho\tau}\int_0^\infty \left\{(m+2)\frac{\sqrt{1+\zeta^2}}{\zeta}+m\frac{\zeta}{\sqrt{1+\zeta^2}}\right\}J_1(\rho\zeta)J_1(\tau\zeta)\,\mathrm{d}\zeta. \tag{89}$$

The closed form expression for the kernel K_1^* in eqn (88) is given by Watson (1966, p. 405). Therefore, from eqns (87)–(89) we have

$$\left.\begin{aligned} & K^*(\rho,\tau) = \frac{(m+2)}{2}\tau\sqrt{\frac{\tau}{\rho}}+k\sqrt{\rho\tau}\left\{m\int_0^\infty b_1(\zeta)J_1(\rho\zeta)J_1(\tau\zeta)\,\mathrm{d}\zeta\right. \\ & \left.\quad +(m+2)\int_0^\infty b_2(\zeta)J_1(\rho\zeta)J_1(\tau\zeta)\,\mathrm{d}\zeta\right\} && 0 < \tau \leqslant \rho \leqslant \omega \\ & K^*(\rho,0) = K^*(0,\rho) = 0 && 0 \leqslant \rho \leqslant \omega \\ & b_1(\zeta) = \frac{\zeta}{\sqrt{1+\zeta^2}} && 0 \leqslant \zeta < \infty \\ & b_2(\zeta) = \frac{\sqrt{1+\zeta^2}}{\zeta} && 0 < \zeta < \infty \end{aligned}\right\}. \tag{90}$$

The first term appearing in the kernel (90) is continuous on its square domain of definition $[0, \omega] \times [0, \omega]$. The auxiliary function b_1 entering eqn (90) takes on positive values and is uniformly bounded on $[0, \infty)$. Further,

$$b_1(\zeta) = o(\zeta) \quad \text{as } \zeta \to 0, \quad b_1(\zeta) = O(\zeta^0) \quad \text{as } \zeta \to \infty. \tag{91}$$

By virtue of eqn (91) and since $J_1(\zeta) = o(\zeta^{-1/2})$ as $\zeta \to \infty$ the first integral inside the curly brackets in eqn (90) is uniformly convergent on the closed square $(0 \leqslant \rho \leqslant \omega, 0 \leqslant \tau \leqslant \omega)$. Therefore, according to the theorem of Stokes (Whittaker and Watson, 1948, p. 73) the first and the second terms of the kernel (90) are bounded on $(0 \leqslant \rho \leqslant \omega, 0 \leqslant \tau \leqslant \omega)$. It only remains to investigate the behavior of the third term appearing in the kernel (90) in the form of an integral. This integral is not uniformly convergent on the closed square $(0 \leqslant \rho \leqslant \omega, 0 \leqslant \tau \leqslant \omega)$. The same integral in the definition of the kernel was also present in the corresponding Fredholm integral equation of the mode-III crack problem (Vardoulakis *et al.*, 1996). It was shown there that is belongs to the Lebesgue class L^2 in the square $[0, \omega] \times [0, \omega]$, thus we have avoided the restrictive hypothesis of continuity. In fact, according to the following definitions

$$\left.\begin{aligned} I(\rho, \tau) &= \sqrt{\rho\tau} \int_0^\infty \frac{\sqrt{1+\zeta^2}}{\zeta} J_1(\rho\zeta) J_1(\omega\zeta)\, d\zeta \\ I_*(\rho, \tau) &= \sqrt{\rho\tau} \int_0^\infty \frac{1}{\zeta} J_1(\rho\zeta) J_1(\omega\zeta)\, d\zeta + \sqrt{\rho\tau} \int_0^\infty J_1(\rho\zeta) J_1(\omega\zeta)\, d\zeta \quad 0 < \tau \leqslant \rho \leqslant \omega \end{aligned}\right\} \tag{92}$$

it is valid that

$$I(\rho, \tau) \leqslant I_*(\rho, \tau) \quad 0 < \tau \leqslant \rho \leqslant \omega. \tag{93}$$

The first integral in the definition of I_* is continuous and bounded in $[0, \omega] \times [0, \omega]$. On the other hand, the second integral in I_* is divergent as $\tau \to \rho$. Indeed, an examination of eqn $(92)_2$ reveals that in the limit as $\tau \to \rho$

$$\begin{aligned} I_*(\rho, \tau) &= \sqrt{\rho\tau} \int_0^\infty J_1(\rho\zeta) J_1(\omega\zeta)\, d\zeta + O(1) \\ &= -\frac{1}{\pi}\sqrt{\frac{\rho}{\tau}} \left[\log\left(1 - \frac{\tau^2}{\rho^2}\right) + O(1)\right] \quad 0 < \tau < \rho \leqslant \omega. \end{aligned} \tag{94}$$

However, by placing I_* in the Lebesgue class L^2 the restrictive hypothesis of continuity (and consequently boundness) can be avoided. The necessary and sufficient condition for the integral $I_*(\rho, \tau)$ and, hence, for $I(\rho, \tau)$ to belong to L^2 is (Tricomi, 1957)

$$\|I\|^2 \leqslant \|I_*\|^2 = \int_0^\omega d\rho \int_0^\omega I^2{}_*(\rho, \tau)\, d\tau < \infty \tag{95}$$

where $\|\cdot\|$ denotes the norm. The validity of the above was explicitly presented in Vardoulakis *et al.* (1996).

Hence, from the preceding investigation it is concluded that if the material length ratio $k = \ell'/\ell \neq 0$ the kernel $K^*(\rho, \tau)$ is an L^2 function in the square $[0, \omega] \times [0, \omega]$, whereas if $k = 0$ the kernel is continuous and bounded on $(0 \leqslant \rho \leqslant \omega, 0 \leqslant \tau \leqslant \omega)$. According to Fredholm's first theorem (Tricomi, 1957; Kanwal, 1971), the solution of the integral (83) exists and it is unique provided that λ is not an eigenvalue of the homogeneous equation associated with eqn (83). The solution of the Fredholm integral equation is written in the particularly compact form (Kanwal, 1971)

$$\Phi(\rho) = R(\rho)+\lambda \int_0^\omega \Gamma(\rho,\tau;\lambda) R(\tau)\,d\tau \quad 0 \leqslant \rho < \omega \tag{96}$$

where R is given by eqn (85) and it is integrable, $\Gamma(\rho,\tau;\lambda)$ is the resolvent kernel of the Fredholm integral eqn (83) given as the quotient of two power series in λ

$$\Gamma(\rho,\tau;\lambda) = \frac{\Delta(\rho,\tau;\lambda)}{\Delta(\lambda)} = \frac{\sum_{p=0}^{\infty} \frac{(-\lambda)^p}{p!} C_p(\rho,\tau)}{\sum_{p=0}^{\infty} \frac{(-\lambda)^p}{p!} c_p} \tag{97}$$

with

$$c_0 = 1, \quad c_p = \int_0^\omega C_{p-1}(s,s)\,ds, \quad C_p(\rho,\tau) = c_p K^*(\rho,\tau) - p\int_0^\omega K^*(\rho,x) C_{p-1}(x,\tau)\,dx. \tag{98}$$

According to Hadamard's theorem (Tricomi, 1957) if the kernel is either bounded or an L^2 function and, of course integrable, both series $\Delta(\rho,\tau;\lambda)$, $\Delta(\lambda)$ in eqn (97) converge for all values of λ, i.e. both series are entire functions of λ. Hence, the resolvent kernel $\Gamma(\rho,\tau;\lambda)$ is a meromorphic function of λ and, according to the generalized Liouville theorem, the resolvent exists for all λ provided $\Delta(\lambda) \neq 0$.

By means of the shifting (or sampling) property of the delta generalized function

$$\int \delta(x-x_0)\psi(x_0)\,dx = \psi(x_0)$$

and the value of the function $R(\rho)$ given by eqn (85), eqn (96) takes the following form

$$\Phi(\rho) = -\frac{(m+2)}{2(m+1)}\frac{\sigma_0}{G}\sqrt{\omega}\delta(\omega-\rho)+\Phi^*(\rho) \quad 0 \leqslant \rho < \omega \tag{99}$$

where the function $\Phi^*(\rho)$ is defined as follows

$$\left.\begin{aligned}
\Phi^*(\rho)/(\sigma_0/G) &= \frac{1}{2(m+1)}\left\{\frac{m}{4}\sqrt{\rho\omega} I_1(\rho) K_1(\omega) + \frac{(m+2)}{2(m+1)}\sqrt{\omega}\Gamma(\rho,\omega;\lambda)\right. \\
&\quad \left. -\frac{m}{8(m+1)}\omega K_1(\omega)\int_0^\omega \sqrt{\tau} I_1(\tau)\Gamma(\rho,\tau;\lambda)\,d\tau\right\} \\
&\quad +\frac{k}{2(m+1)}\left\{-\frac{(m+2)}{2}\frac{\rho\sqrt{\rho}}{\omega}\,{}_2F_1\left(\frac{3}{2},\frac{1}{2};2;\frac{\rho^2}{\omega^2}\right)\right. \\
&\quad +\frac{1}{2\pi}\sqrt{\omega}\left[Q_{1/2}\left[\frac{\rho^2+\omega^2+3.6}{2\rho\omega}\right]+1.9Q_{1/2}\left[\frac{\rho^2+\omega^2+1}{2\rho\omega}\right]\right] \\
&\quad +\frac{(m+2)}{4(m+1)}\frac{1}{\omega}\int_0^\omega \tau\sqrt{\tau}\,{}_2F_1\left(\frac{3}{2},\frac{1}{2};2;\frac{\tau^2}{\omega^2}\right)\Gamma(\rho,\tau;\lambda)\,d\tau \\
&\quad -\frac{1}{4(m+1)}\frac{\sqrt{\omega}}{\pi}\int_0^\omega\left[2Q_{1/2}\left[\frac{\tau^2+\omega^2+3.6}{2\tau\omega}\right]\right. \\
&\quad \left.\left. +1.85mQ_{1/2}\left[\frac{\tau^2+\omega^2+1}{2\tau\omega}\right]\right]\Gamma(\rho,\tau;\lambda)\,d\tau\right\} \quad 0 \leqslant \rho < \omega
\end{aligned}\right\}. \tag{100}$$

5. SOLUTION NEAR THE TIP AND ENERGY RELEASE RATE

By using again the shifting property of the delta function, and recalling eqns (55) and (60), as well as the scaling relation (63) and transformation formula (82), it can be demonstrated that the leading term of the representation (99) of the function $\Phi(\rho)$ cancels out the displacements predicted by the classical theory. Thus, the shape of the crack is no longer elliptical, as predicted by LEFM, but it is given by the formula

$$v(x,0^-) = \ell\int_{\chi}^{\alpha}\psi^*(\rho)\sqrt{\rho^2-\chi^2}\,\mathrm{d}\rho \tag{101}$$

where $\psi^*(\rho) = \Phi^*(\rho)/\sqrt{\rho}$.

After integration by parts of expression (101) and by an asymptotic analysis of the solution close to the crack tip, we obtain (Vardoulakis *et al.*, 1996)

$$v(r,0^+) = \frac{2\sqrt{2}}{3}\alpha^{1/2}\psi^*(\alpha-\eta)r^{3/2}+\mathrm{o}(r^{5/2})+\varepsilon(\eta) \quad \text{as } r\to 0 \tag{102}$$

where we have switched to the state before the transformations (63) and $r = \alpha - x$. In eqn (102), η is a small length with respect to the semi-crack length α, in order to remove the weak logarithmic singularity of ψ^* at $t = \alpha$. The following upper bound may be found which gives the desired accuracy as a function of η

$$|\varepsilon(\eta)| = \left|\int_{\alpha-\eta}^{\alpha}\psi^*(t)\sqrt{t^2-x^2}\,\mathrm{d}t\right| \leqslant N(\eta^*), \quad \eta^* = \frac{1}{\pi}\frac{\eta}{\alpha},$$

$$N(\eta^*) = \frac{(m+2)}{2(m+1)}\frac{\sigma_0}{G}\frac{k\alpha^2}{\ell}|-0.15\eta^*+\eta^*\log\eta^*| \to 0 \quad \text{as } \eta^*\to 0. \tag{103}$$

According to eqn (102) the mode-I crack shape predicted by gradient elasticity with surface energy is described by the equation

$$\left(\frac{x}{\alpha}\right)^2+\left(\frac{v(x,0^-)}{b}\right)^{2/3} = 1 \tag{104}$$

where $b = v(0,0^-)$. That is, the crack lips form a cusp of the first kind with zero enclosed angle and zero first derivative of the displacement at the crack tip, as it is shown in Fig. 4.

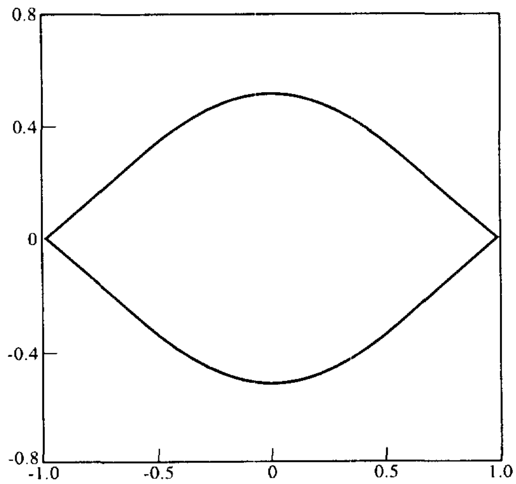

Fig. 4. Crack with tips in the form of cusps of first kind ($\alpha = b = 1$).

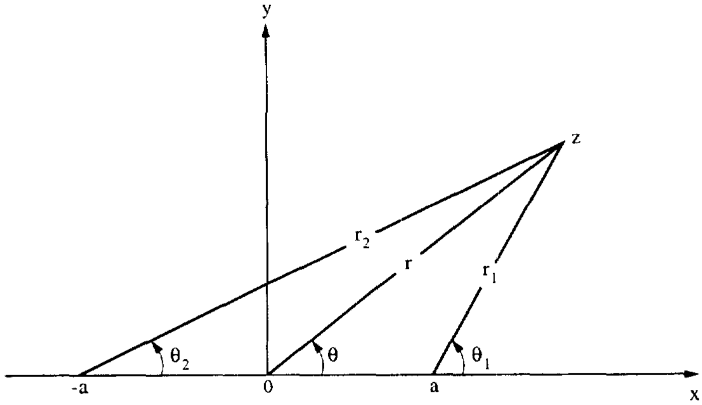

Fig. 5. Crack and coordinates.

It is noted that LEFM predicts an infinite displacement gradient at the crack tip. This fact indicates that the present gradient elasticity theory with surface energy predicts the same crack shape as Barenblatt's (1962) "cohesive-zone" theory, but without requiring an extra assumption on the existence of the interatomic forces at the outset beyond those implied by the gradient terms in the generalized constitutive equation.

Formula (52) indicates that the stresses are singular at the crack tip, as in the classical LEFM theory. By applying the well-known Irwin's condition at the crack tip to obtain the stress intensity factor (SIF)

$$K_{\mathrm{I}} = \lim_{x \to \alpha^+} \{\sqrt{2\pi(x-\alpha)}\sigma_{yy}(x, 0^+)\} \tag{105}$$

and the relation (Gradshteyn and Ryzhik, 1980)

$$\int_0^\infty J_1(\alpha\xi)\cos x\xi \,\mathrm{d}\xi = -\frac{\alpha}{\sqrt{x^2-\alpha^2}(x+\sqrt{x^2-\alpha^2})} \quad x > \alpha \tag{106}$$

we find

$$K_{\mathrm{I}} = \sigma_0\sqrt{\pi\alpha} \tag{107}$$

which agrees with the well-known LEFM result.

Let $\bar{S}$ from here on stand for the open half-plane $y \geqslant 0$ together with its bounding edge, i.e. for the region $(0 \leqslant y < \infty, -\infty < x < \infty)$. Introducing polar coordinates r_1, θ_1, r, θ, r_2, θ_2 (Fig. 5) through the relations

$$z = r\,\mathrm{e}^{i\theta}, \quad z-\alpha = r_1\,\mathrm{e}^{i\theta_1}, \quad z+\alpha = r_2\,\mathrm{e}^{i\theta_2} \tag{108}$$

we seek to determine the behavior of the solution at the (singular) endpoints of the crack. For this purpose it is expedient to adopt the notation (Muki and Sternberg, 1965)

$$f(x,y) = \int_0^\infty \tilde{f}(x,y;\xi)\,\mathrm{d}\xi \tag{109}$$

where $f(x,y)$ stands for quantities such as displacements, strains or double stresses; since the present gradient elasticity theory predicts the same stresses as the classical LEFM theory we do not need to find the stress singularity at the crack tips. Since $(\tilde{u}, \tilde{v})$, $\tilde{\varepsilon}_{ij}$, $\tilde{\mu}_{ijk}$ $(i,j = x,y)$ are finite and continuous on $\bar{S}$ for every fixed $\xi \geqslant 0$, it is clear that any possible divergence at the crack tips must stem from the behavior of $(\tilde{u}(\pm\alpha,0;\xi)$, $\tilde{v}(\pm\alpha,0;\xi))$,

$\tilde{\varepsilon}_{ij}(\pm\alpha, 0; \xi)$, $\tilde{\mu}_{ijk}(\pm\alpha, 0; \xi)$) as $\xi \to \infty$; specifically, the singularities in question must be contributed by those portions of the integrands that at $x = \pm\alpha$, $y = 0$ are $O(\xi^{-1})$, or of a larger order of magnitude as $\xi \to \infty$. The foregoing contributions may then be determined in closed form by means of familiar Bessel integral-identities (Watson, 1966; Gradshteyn and Ryzhik, 1980) and by using the following elementary expansion

$$\ell a(\xi) = \xi\ell + \frac{1}{2\xi\ell} - \frac{1}{8\xi^3\ell^3} + o(\xi^{-5}) \tag{110}$$

which follows from eqn (36) and is valid for every fixed positive ℓ as $\xi \to \infty$. Returning to the state before the introduction of non-dimensional variables given by eqn (63), this process yields the following estimates, which hold true as $r_1 \to 0$ for every fixed positive ℓ for the:

(A) *Displacements*

$$\begin{aligned} u(r_1, \theta_1) &= \frac{\sigma_0}{G}\sqrt{\frac{\alpha}{2}}\, r_1^{1/2} \cos\frac{\theta_1}{2}\left[(1+2\nu) - \sin^2\frac{\theta_1}{2}\right] \\ &\quad + \frac{4}{3}\frac{\sigma_0}{G}\sqrt{\frac{\alpha}{2}}\,\hat{\psi}(\alpha-\eta) r_1^{3/2} \cos\frac{3\theta_1}{2} + o(r_1^{5/2}) \\ v(r_1, \theta_1) &= \frac{\sigma_0}{G}\sqrt{\frac{\alpha}{2}}\, r_1^{1/2} \sin\frac{\theta_1}{2}\cos^2\frac{\theta_1}{2} - \frac{\sigma_0}{G}\left[\int_0^{\alpha} \hat{\psi}(t)\,\mathrm{d}t\right] r_1 \sin\theta_1 \\ &\quad - \frac{4}{3}\frac{\sigma_0}{G}\sqrt{\frac{\alpha}{2}}\,\hat{\psi}(\alpha-\eta) r_1^{3/2} \sin\frac{3\theta_1}{2} + o(r_1^{5/2}). \end{aligned} \tag{111}$$

(B) *Stresses*

$$\begin{aligned} \sigma_{xx}(r_1, \theta_1) &= \frac{K_I}{\sqrt{2\pi}} r_1^{-1/2} \cos\frac{\theta_1}{s}\left(1 - \sin\frac{\theta_1}{2}\sin\frac{3\theta_1}{2}\right) + o(r_1^{1/2}) \\ \sigma_{yy}(r_1, \theta_1) &= \frac{K_I}{\sqrt{2\pi}} r_1^{-1/2} \cos\frac{\theta_1}{2}\left(1 + \sin\frac{\theta_1}{2}\sin\frac{3\theta_1}{2}\right) - \sigma_0 + o(r_1^{1/2}) \\ \sigma_{xy}(r_1, \theta_1) &= \frac{K_I}{\sqrt{2\pi}} r_1^{-1/2} \cos\frac{\theta_1}{2}\sin\frac{\theta_1}{2}\cos\frac{3\theta_1}{2} + o(r_1^{1/2}). \end{aligned} \tag{112}$$

(C) *Strains*

$$\begin{aligned} \varepsilon_{xx}(r_1, \theta_1) &= \frac{\sigma_0}{2G}\sqrt{\frac{\alpha}{2}}\, r_1^{-1/2} \cos\frac{\theta_1}{2}\left(1 + \sin\frac{\theta_1}{2}\sin\frac{3\theta_1}{2}\right) \\ &\quad + 2\frac{\sigma_0}{G}\sqrt{\frac{\alpha}{2}}\,\hat{\psi}(\alpha-\eta) r_1^{1/2} \cos\frac{\theta_1}{2} + o(r_1^{3/2}) \\ \varepsilon_{yy}(r_1, \theta_1) &= \frac{\sigma_0}{2G}\sqrt{\frac{\alpha}{2}}\, r_1^{-1/2} \cos\frac{\theta_1}{2}\left(1 - \sin\frac{\theta_1}{2}\sin\frac{3\theta_1}{2}\right) \\ &\quad + 2\frac{\sigma_0}{G}\sqrt{\frac{\alpha}{2}}\,\hat{\psi}(\alpha-\eta) r_1^{1/2} \cos\frac{\theta_1}{2} + o(r_1^{3/2}) \end{aligned}$$

$$\varepsilon_{xy}(r_1,\theta_1) = \frac{\sigma_0}{2G}\sqrt{\frac{\alpha}{2}}r_1^{-1/2}\sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}\cos\frac{3\theta_1}{2} + o(r_1^{3/2}). \tag{113}$$

(D) *Double stresses (herein we record only μ_{yyy}, μ_{yyx})*

$$\mu_{yyx}(r_1,\theta_1) = \frac{3}{2}\sigma_0\sqrt{\frac{\alpha}{2}}\ell^2 r_1^{-3/2}\sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}\sin\frac{5\theta_1}{2}$$

$$-\sigma_0\sqrt{\frac{\alpha}{2}}\ell' r_1^{-1/2}\sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}\cos\frac{3\theta_1}{2} + o(r_1^{1/2})$$

$$\mu_{yyy}(r_1,\theta_1) = -\sigma_0\sqrt{\frac{\alpha}{2}}\ell^2\left\{\left[\frac{2}{1-2\nu}\sin\frac{3\theta_1}{2} + 3\sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}\cos\frac{5\theta_1}{2}\right]r_1^{-3/2}\right.$$

$$\left. + \frac{2}{1-2\nu}\hat{\psi}(\alpha-\eta)r_1^{-1/2}\sin\frac{\theta_1}{2}\right\}$$

$$-\sigma_0\sqrt{\frac{\alpha}{2}}\ell' r_1^{-1/2}\cos\frac{\theta_1}{2}\left(\frac{1}{1-2\nu} - \sin\frac{\theta_1}{2}\sin\frac{3\theta_1}{2}\right) + o(r_1^{1/2}) \tag{114}$$

where we have set $\hat{\psi}(\alpha-\eta) = \psi^*(\alpha-\eta)/(\sigma_0/G)$.

It is apparent from eqns (111)–(114) that in the limit as $r_Y \to 0$

$$u(x,y) = O(1),\ v(x,y) = O(1),\ \sigma_{\alpha\beta}(x,y) = O(r_1^{-1/2}),\ \varepsilon_{\alpha\beta}(x,y) = O(r_Y^{-1/2}),$$

$$\mu_{\alpha\beta\gamma}(x,y) = O(r_Y^{-3/2}),\ \alpha,\beta,\gamma = x,y,\ Y = 1,2. \tag{115}$$

Equations (111) show that the displacements associated with mode-I deformation remain bounded at the crack tip in both the modified and classical solutions. It is noted that we have obtained values of u and v that involve symmetric and anti-symmetric forms of $\theta/2$ and so they do indeed describe opening mode deformations. From eqns (111) it is also noted that the present gradient elasticity theory predicts that the part of the normalized displacement $v(r_1,\theta_1)/(\sigma_0/G)$ that is of the order of $O(r_Y^{1/2})$ does not involve Poisson's ratio, in contrast to LEFM. It is important to observe that, while the order of the strain singularities in eqn (113) is the same with that predicted by LEFM, i.e. $O(r_Y^{-1/2})$, the detailed structure of this singularity is different (except in the case of the shear strain ε_{xy}). In contrast to LEFM, the dominating normalized strains $\varepsilon_{\alpha\beta}(r_1,\theta_1)/(\sigma_0/G)$ near the tip that are predicted by the present gradient elasticity theory do not depend on Poisson's ratio. This is due to the fact that the term $yB_2(\xi)\,e^{-y|\xi|}$ in the representation for the normal crack displacement (33), which is responsible for an inverse square root singularity in the strains, is not eliminated by the gradient elasticity. The first derivative of v with respect to x near the crack tip can be found from $(111)_2$ to be

$$\frac{\partial v}{\partial x}(r_1,\theta_1) = \frac{\sigma_0}{G}\sqrt{\frac{\alpha}{2}}r_1^{-1/2}\sin\theta_1\cos\frac{\theta_1}{2}\left\{\frac{1}{4}\cos\theta_1 - \frac{3}{2}\cos^2\frac{\theta_1}{2} - 1\right\}$$

$$-\frac{\sigma_0}{G}\sqrt{2\alpha}r_1^{1/2}\hat{\psi}(\alpha-\eta)\left\{\cos\theta_1\sin\frac{3\theta_1}{2} + \sin\theta_1\cos\frac{3\theta_1}{2}\right\} + o\ \ (1).$$

Hence, the present gradient elasticity theory predicts that the slope of the opening displacement on the crack plane ($y = 0$) with respect to x is continuous, i.e.

$$\lim_{x \to \alpha^-} \frac{\partial v}{\partial x}(x, 0^+) = \lim_{x \to \alpha^+} \frac{\partial v}{\partial x}(x, 0^+) = 0$$

and an undesirable result which is given by LEFM is removed. It is worth noting that Barenblatt's "cohesive-zone" theory predicts infinite slope of the crack opening displacement at the physical crack–cohesive zone tip, even though a smooth closure of the crack faces is assured.

It is also worth noticing from eqn $(114)_2$ that $\mu_{yyy}(r_1, \pi) \neq 0$ as $r_1 \to 0$. This unexpected behavior of the double stress may be attributed to the boundary layer phenomenon due to the presence of the term ℓ^2 in the operator $\bar{D}^2$ defined by eqn (21). This phenomenon indicates that the main difference between elasticity and gradient dependent elasticity lies in a boundary layer. An elasticity solution is, but the interior solution of an associated gradient dependent elasticity problem. Far from the crack tip, that is, for $r_1 \gg \ell$ it is valid that the double stress $\mu_{yyy}(r_1, \pi)$ vanishes. This particular effect led us, in turn, to define the double stress intensity factor that acts behind the crack tip as follows

$$K_\mu = \sqrt{2\pi} \lim_{r_1 \to 0^-} r_1^{3/2} \mu_{yyy}(r_1, \pi). \tag{116}$$

Then from eqn $(114)_2$ it may be found

$$K_\mu = \frac{2\ell^2}{1-2v} \sigma_0 \sqrt{\pi\alpha}. \tag{117}$$

Since $\ell^2 > 0$ the above formula predicts the very important result that behind the crack tip tensile or "cohesive" double forces act tending to bring the two opposite crack lips in close contact, thus leading to crack cusping (Fig. 4) and stiffening. The magnitude of these cohesive double forces increases with the square of ℓ. From the above it is evident that the volumetric strain–gradient parameter ℓ accounts for the internal forces in the volume of the material which resist deformation and fracture.

Furthermore, on defining the double stress intensity factor that acts in front of the crack tip as

$$K_\mu^* = \sqrt{2\pi} \lim_{r_1 \to 0^+} r_1^{1/2} \mu_{yyy}(r_1, 0) \tag{118}$$

then from eqn $(114)_2$ we find

$$K_\mu^* = -\frac{\ell'}{1-2v} \sigma_0 \sqrt{\pi\alpha}. \tag{119}$$

It is interesting to note from the above formula, that for negative values of the surface energy parameter ℓ' the double stress intensity factor K_μ^* becomes positive, thus leading to near-tip stress amplification. On the other hand, for positive ℓ'-values the double stress intensity factor takes negative values, leading to near-tip stress shielding.

The energy release during an infinitesimal advancement of the crack tip by a distance $\delta\alpha$ is given in polar coordinates by

$$\delta U = \int_0^{\delta\alpha} \sigma_{yy}(\delta\alpha - h, 0^+) v(h, \pi) \, \mathrm{d}h. \tag{120}$$

By inserting into eqn (120) the values of v and σ_{yy} as they are given by eqns $(111)_2$ and $(112)_2$, respectively, and carrying out the integration we find

$$\delta U = K_I k_I \delta\alpha^2 \tag{121}$$

where we have set

$$k_I = \frac{\sigma_0\sqrt{\pi\alpha}}{4G}\psi^*(\alpha-\eta) = \frac{K_I}{4G}\hat{\psi}(\alpha-\eta). \tag{122}$$

In view of eqn (121) and Griffith's rupture criterion (Griffith, 1921)

$$\delta U \geqslant 2\gamma\delta\alpha \tag{123}$$

where 2γ [FL^{-1}] is the so-called "specific fracture energy", we obtain the following inequality which involves the important physical quantity of the energy release rate G_I in mode-I crack propagation

$$G_I = \frac{\delta U}{\delta\alpha} = K_I k_I \delta\alpha \geqslant 2\gamma. \tag{124}$$

If quantity γ is independent of the crack advancement $\delta\alpha$, then the left-hand part of inequality (124) goes to zero and the gradient elasticity theory predicts that there is no contribution to the work rate from the "holding force" on the crack extension. Since the latter is not possible by fundamental physical conditions, γ has to depend linearly on $\delta\alpha$ for crack tip propagation distances that are not large as compared to the grain size of the brittle material, that is,

$$\gamma = \gamma(\delta\alpha) = \beta\delta\alpha \quad \text{as } \delta\alpha \to 0 \tag{125}$$

where the quantity β has the dimensions of specific volume energy [FL^{-2}], called hereafter "modulus of cohesion". Definition (125) is in agreement with the experimental results of Hoagland *et al.* (1973), who found that the specific fracture energy or fracture resistance of Salem limestone was an increasing function of crack propagation distance at an early stage of crack extension, but finally reached asymptotically a constant value corresponding to large, relative to the grain size, pre-existing flaws in the rock. γ-curve, i.e. the curve of γ as a function of $\delta\alpha$, must start from zero as indicated in eqn (125); at zero stress the size of the process zone is zero—it requires no energy to form a process or microcracking zone of zero size. By virtue of definition (125), Griffith's rupture criterion (124) is modified as follows

$$\Psi(\alpha-\eta;\ell,\ell') \geqslant \beta, \quad \Psi(\alpha-\eta;\ell,\ell') = \frac{\pi\alpha}{8}\frac{\sigma_0^2}{G}\hat{\psi}(\alpha-\eta) = \frac{K_I^2}{8G}\hat{\psi}(\alpha-\eta) \tag{126}$$

where the function Ψ in eqn (126) depends on the applied pressure on the faces of the crack, on crack length and on material length parameters ℓ, ℓ'. The corresponding critical value of K_I which represents the fracture resistance of the material is denoted by K_{IC} and is called "fracture toughness" or "critical stress intensity factor". Note from eqn (126) that

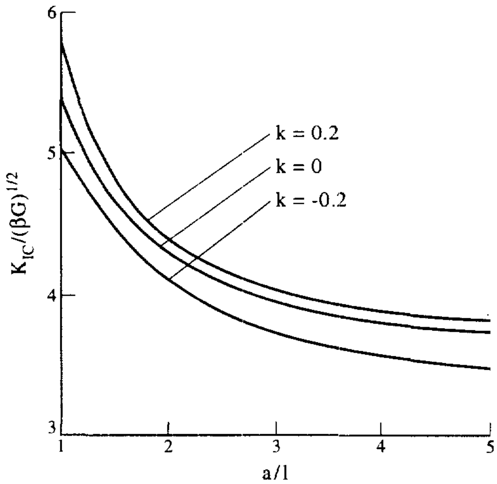

Fig. 6. Size effect of the normalized K_{IC} for three values of the material length ratio $k = \ell'/\ell$ and for Poisson's ratio of the material $\nu = 1/4$.

$$K_{IC} = \sqrt{\frac{8\beta G}{\psi(\alpha-\eta)}}. \tag{127}$$

The variation of the normalized K_{IC} with the ratio α/ℓ is depicted in Fig. 6 for three values of the material length ratio, namely for $k = -0.2$, 0 and 0.2, and for Poisson's ratio $\nu = 0.25$. It can be seen that the resistance to fracture of the material decreases with decreasing volumetric energy parameter ℓ as it is also predicted by eqn (117); positive values of the surface energy parameter ℓ' further enhance the strength of the material, whereas negative values of the surface energy parameter lead to a decrease of the fracture toughness of the material. Hence, the interpretation of relations (117) and (119) given previously, is confirmed by the results give in Fig. 6. Notice that LEFM does not predict an effect of the size of the crack on K_{IC}, that is, it considered K_{IC} as a constant.

If the classical fracture criterion

$$\frac{(1-\nu^2)}{E}K_I^2 \geqslant 2\gamma$$

were used to compute the critical energy release rate G_{IC} from the critical stress intensity factor K_{IC}, say, for a rock, the value of G_{IC} so determined would be several orders of magnitude greater than the surface energy 2γ. Or, conversely, if G_{IC} were equated to the surface energy, then unrealistically small failure loads would be predicted for rocks and ceramics. On the other hand, the modified fracture criterion (126) predicts reasonable surface energy and failure stress values for appropriate values of the parameters ℓ, ℓ', β. Performing mode-I fracture mechanics experiments on brittle specimens with known shear modulus G one then can estimate the critical K_{IC} and the modulus of cohesion β from the slope of the specific fracture energy 2γ against crack length. Accordingly, the characteristic material lengths ℓ, ℓ' can be estimated from the size effect exhibited by K_{IC} (Fig. 6).

From condition (126) for the onset of crack extension a first-order approximation of the breaking stress σ_t for $\ell' = 0$ can be obtained as follows

$$\sigma_t \cong \sqrt{\frac{8E\beta}{\pi(1-\nu^2)}\frac{\ell}{\alpha}}.$$

The above inverse first-power dependence of strength on the size of the crack-like defect agrees with experimental results on elastomers presented by Bueche and Berry (1959). On the other hand, Griffith's criterion predicts an inverse square-root relation, so it does not give the correct dependence of the tensile strength on the size of the pre-existing crack. It is to be expected that the above modified rupture criterion proposed herein, will not apply quantitatively to rubbery materials, but dimensional requirements indicate that the above dependence of strength on cut size should be a good approximation.

6. NUMERICAL RESULTS AND DISCUSSION

We proceed now to the derivation of further numerical results. The two quantities of primary physical interest are the transverse displacement v at the center of the crack and the energy release rate associated with an incremental crack extension along the crack-axis: the first supplies a measure of the deformations; the second is indicative of the magnitude of the crack driving force.

The improper integral representation for the symmetric kernel K_2^* appearing in eqn (89) is inconvenient for numerical purposes because of the infinite range of integration and the oscillatory character of the integrand concerned. An alternative representation, which is free from these deficiencies is readily deduced by means of suitable contour integration, the details of which is given in Appendix B. The result obtained in this manner, which was employed in the numerical evaluations carried out has the following form

$$K_2^*(\rho,\tau) = \begin{cases} \dfrac{2\sqrt{\rho\tau}}{\pi}\displaystyle\int_0^1 \left\{(m+2)\sqrt{1-X^2}+\dfrac{m}{\sqrt{1-X^2}}\right\} X^{-2}K_1(\rho/X)I_1(\tau/X)\,\mathrm{d}X \\ \quad +\dfrac{(m+2)}{2}\sqrt{\rho\tau}\dfrac{\tau}{\rho} \quad 0<\tau\leqslant\rho\leqslant\omega \\ \dfrac{2\sqrt{\rho\tau}}{\pi}\displaystyle\int_0^1 \left\{(m+2)\sqrt{1-X^2}+\dfrac{m}{\sqrt{1-X^2}}\right\} X^{-2}K_1(\tau/X)I_1(\rho/X)\,\mathrm{d}X \\ \quad +\dfrac{(m+2)}{2}\sqrt{\rho\tau}\dfrac{\rho}{\tau}, \quad 0<\rho\leqslant\tau\leqslant\omega. \end{cases} \tag{128}$$

The energy release rate is computed numerically by using the formula (Vardoulakis *et al.*, 1996)

$$G_1 = \sigma_0\frac{\partial}{\partial\alpha}\left[\int_0^\alpha v(x,0)\,\mathrm{d}x\right] = \frac{\pi}{4}\frac{\sigma_0^2}{G}\ell\frac{\partial}{\partial\omega}\int_0^\omega \rho^2\hat{\psi}(\rho)\,\mathrm{d}\rho, \quad \hat{\psi}(\rho) = \psi^*(\rho)/(\sigma_0/G). \tag{129}$$

All the integrals are evaluated by a 20-point Gauss–Legendre numerical quadrature scheme. For convenience, the values of the ratio σ_0/G is taken equal to one and the semi-crack length α is chosen to be our unit of length, that is $\alpha = 1$. In order to demonstrate the nice convergence behavior of the transverse displacement at the mid-point of the crack $v(0,0)$ given by (101), for increasing number of terms in the truncated series (97) and for various values of the relative volume energy parameters ℓ/α and for $\ell' = 0$, Table 1 is constructed.

Table 1. Effect of the relative volume energy parameter ℓ/α on the convergency characteristics of the transverse displacement $v(0,0)$ at center of mode-I crack for $\nu = 1/4$ and $\ell'/\alpha = 0$

				ℓ/α			
p	0.1	0.2	0.3	0.4	0.5	0.6	0.7
0	12.019810	3.042537	1.374387	0.785491	0.509756	0.358145	0.265671
1	4.0069280	1.296227	0.734407	0.496408	0.362107	0.275899	0.216693
2	2.0861190	0.879052	0.599989	0.446853	0.341972	0.266983	0.212430
3	1.3608770	0.759404	0.575229	0.440985	0.340363	0.266476	0.212250
4	1.0482000	0.732197	0.572407	0.440603	0.340296	—	—
5	0.9227681	0.732197	0.572225	0.440589	—	—	—
6	0.8837693	0.757929	0.572218	—	—	—	—
7	0.8861250	0.727920	—	—	—	—	—

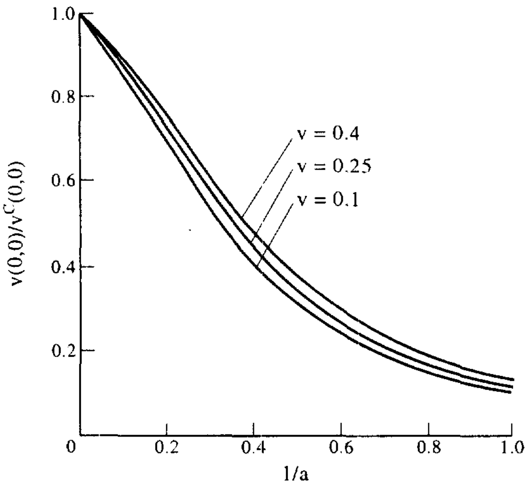

Fig. 7. Transverse displacement $v(0,0)$ at center of mode-I crack for $\ell'/\alpha = 0$.

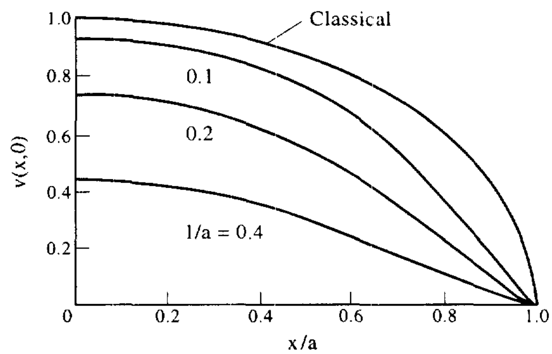

Fig. 8. Shape of mode-I crack for $v = 1/4$ and $\ell'/\alpha = 0$.

Furthermore, Fig. 7 depicts the ratio of the transverse displacement at the center of the crack to the corresponding classical value $v^c(0,0)$ as a function of ℓ/α with zero value of the surface energy parameter, and for the three values of Poisson's ratio at hand. From this figure it is noted that the solution of the Fredholm equation definitely tends to $v(0,0)/v^c(0,0) = 1$ for $\ell/\alpha \to 0$; thus the transition to the classical theory is continuous as far as the displacement under consideration is concerned. As is apparent, the opening of the crack at its midpoint diminishes monotonically in the departure from the classical theory (shielding effect), i.e. as ℓ/α increases, this decrease being slightly more pronounced at smaller values of Poisson's ratio.

Figure 8 displays the upper-right quarter of the crack shape obtained from the present gradient elasticity theory for $\ell/\alpha = 0.1, 0.2, 0.4$, $\ell' = 0$ and $v = 0.25$. In the same figure, the classical elliptic crack profile for the same value of the Poisson ratio is also shown. As is apparent from this figure the gradient elasticity theory predicts cusping crack tips with zero first derivative of the transverse displacement at this region. The enhancement of crack displacements (material degradation effect) for negative values of the material length ratio $k = \ell'/\ell$ and for $\ell/\alpha = 0.4$, $v = 0.25$ is displayed in Fig. 9. As it is also shown in the same figure the crack stiffening effect (toughening mechanism) for positive values of k is less pronounced compared to that of negative k. It seems that the crack stiffening effect, or material stiffening effect, is mainly controlled by the volumetric strain–gradient term (see Table 1 and Figs 7 and 8) whereas the crack compliance effect, or material degradation effect, is controlled by the surface energy term (Fig. 9). Thus, the resulting enhancement or degradation of the toughness of the crack is ultimately dependent upon the net outcome of the material lengths ℓ, ℓ'. This is a welcome property of the mathematical model as far as the description of the experimental data is concerned.

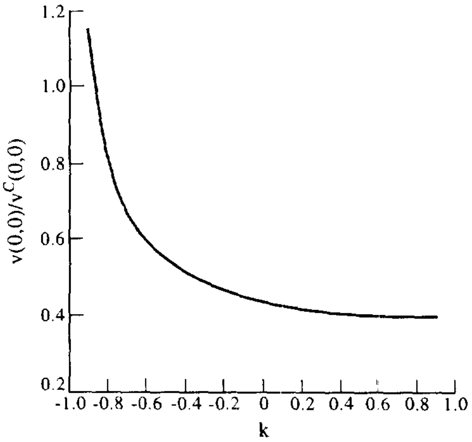

Fig. 9. Effect of material length ratio $k = \ell'/\ell$ on the transverse displacement $v(0,0)$ at center of mode-I crack for $\nu = 1/4$ and $\ell/\alpha = 0.4$.

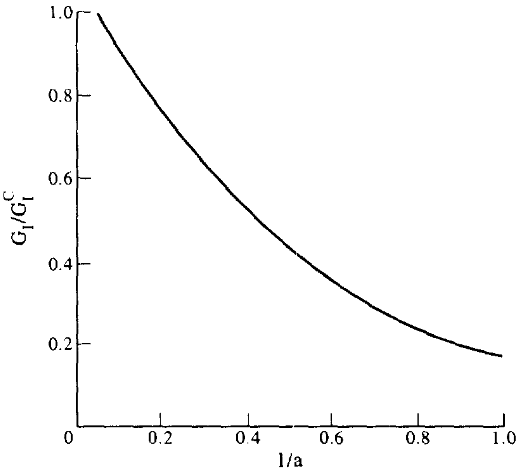

Fig. 10. Effect of the relative volume energy parameter ℓ/α on the dimensionless energy release rate of the pressurized mode-I crack for $\nu = 1/4$ and $\ell'/\alpha = 0$.

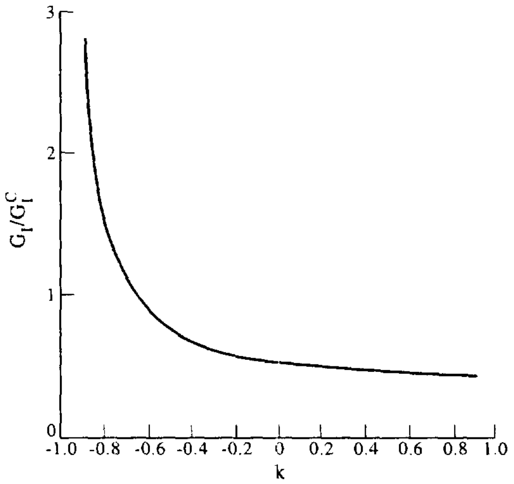

Fig. 11. Effect of the material length ratio $k = \ell'/\ell$ on the dimensionless energy release rate of the pressurized mode-I crack for $\nu = 1/4$ and $\ell/\alpha = 0.4$.

As is expected the crack stiffening effect is leading to lower energy release rate or crack driving force G_I compared to that predicted by LEFM. This is clearly demonstrated in Fig. 10 where it can be seen that the dimensionless ratio G_I/G_I^c, with $G_I^c = (1-\nu)K_I^2/2G$ to be the classical value of the energy release rate in mode-I crack deformation, decreases monotonically as the ratio ℓ/α increases (for $\nu = 0.25, \ell' = 0$). From the same figure it is observed that G_I/G_I^c definitely tends to the value of 1 for $\ell/\alpha \to 0$. On the other hand, the energy release rate amplification effect for negative values of the material length ratio k and for $\ell/\alpha = 0.4$, $\nu = 0.25$, is shown in Fig. 11. In the same figure the shielding effect for $k > 0$ is also shown. As in the case of displacements, the shielding effect on the energy release rate for positive surface energy parameter is much less pronounced as compared to the amplification effect of negative surface energy parameter. From the above results it is clear that the presented gradient elasticity theory with surface energy, in conjunction with properly designed experiments provides a useful quantitative design tool for insight into main crack–microdefect interaction phenomena in brittle materials.

Acknowledgements—The writer would like to thank Professor I. Vardoulakis for suggesting this problem and for valuable discussions in the course of the calculations. The author wishes also to thank the anonymous referee for his/her helpful remarks.

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APPENDIX A

We show here how the integral eqn (73) is obtained from eqn (72). First, returning to the state before the introduction of the dimensionless variables defined by eqn (63), eqn (72) can be rewritten as follows

$$\psi(t) = -\frac{1}{2(m+1)}\frac{2t}{\pi}\int_t^{\infty}\frac{1}{\sqrt{x^2-t^2}}\frac{\mathrm{d}}{\mathrm{d}x}[f(x)]\,\mathrm{d}x. \tag{A1}$$

Then, by taking into account the well-known identities (Gradshteyn and Ryzhik, 1980)

$$\int_0^{\infty}\frac{\sin(yx)\,\mathrm{d}x}{\sqrt{x^2-t^2}} = \frac{\pi}{2}J_0(yt),\quad \int_0^{\xi} yJ_0(ty)\,\mathrm{d}y = \frac{\xi}{t}J_1(t\xi),\ \int y\sin y\,\mathrm{d}y = \sin y - y\cos y \tag{A2}$$

it is found that

$$\int_0^{\infty}\frac{1}{\sqrt{x^2-t^2}}\frac{\mathrm{d}}{\mathrm{d}x}[\gamma(x\xi)]\,\mathrm{d}x = -\frac{\pi}{2t}J_1(t\xi). \tag{A3}$$

By inserting into the differential operator of (A1) the first terms of the function $F(x)/x$ as they are given by eqn (66) one obtains

$$I_1 = -\frac{1}{2(m+1)}\frac{2t}{\pi}\int_t^{\infty}\frac{1}{\sqrt{x^2-t^2}}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sin(x\xi)}{\xi}\left(-\frac{\sigma_0}{2G}\ell^{-1}\alpha\right)\int_0^{\infty}\xi\left[2\left(2+m+\frac{2k}{\xi}\right)J_1(\alpha\xi)\,\mathrm{d}\xi\right]\mathrm{d}x. \tag{A4}$$

Then by interchanging the order of the integrations entering in eqn (A4) and by virtue of eqn (A3), as well as the values of the integrals (Gradshteyn and Ryzhik, 1980)

$$\left.\begin{aligned}\int_0^{\infty}\xi J_1(\alpha\xi)J_1(t\xi)\,\mathrm{d}\xi &= \frac{\delta(\alpha-t)}{(\alpha t)^{1/2}}\\ \int_0^{\infty}J_1(\alpha\xi)J_1(t\xi)\,\mathrm{d}\xi &= \frac{t}{2\alpha^2}\,{}_2F_1\left(\frac{3}{2},\frac{1}{2};2;\frac{t^2}{\alpha^2}\right)\end{aligned}\right\} \tag{A5}$$

where $\delta(\cdot)$ is the generalized delta function of Dirac and ${}_2F_1(a,b;c;z)$ is Gauss's hypergeometric function, we obtain the result

$$I_1 = -\frac{1}{4(m+1)}\frac{\sigma_0 \ell^{-1}\alpha}{G}\left[2(2+m)\frac{\delta(\alpha-t)}{(\alpha t)^{1/2}} + \frac{kt}{\alpha^2}\,{}_2F_1\left(\frac{3}{2},\frac{1}{2};2;\frac{t^2}{\alpha^2}\right)\right]. \tag{A6}$$

In a second step, we introduce into the differential operator of (A1) the remaining terms of the function $F(x)/x$ as they are given by eqn (66), that is

$$I_2 = -\frac{1}{2(m+1)}\frac{2t}{\pi}\int_t^{\alpha}\frac{1}{\sqrt{x^2-t^2}}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sin(x\xi)}{\xi}\left(-\frac{\sigma_0}{2G}\ell^{-1}\alpha\right)\int_0^{\infty}\xi\left\{\frac{2(m+2)k}{\sqrt{1+\xi^2}} + 4\xi(m\xi-k) - \frac{\xi}{\sqrt{1+\xi^2}}(4m\xi^2+2m-4k\xi)\right\}J_1(\alpha\xi)\,\mathrm{d}\xi\right]\mathrm{d}x. \tag{A7}$$

In a similar fashion with that used to deduce relation (A6) we derive

$$I_2 = -\frac{1}{4(m+1)}\frac{\sigma_0}{G}\ell^{-1}\alpha\int_0^{\infty}\left\{\frac{2\xi}{\sqrt{1+\xi^2}}[(m+2)k-m\xi] + \frac{\xi(\sqrt{1+\xi^2}-\xi)(4m\xi^2-4k\xi)}{\sqrt{1+\xi^2}}\right\}J_1(t\xi)J_1(\alpha\xi)\,\mathrm{d}\xi. \tag{A8}$$

Finally, after introducing the integral in eqn (69) into the differential operator of (A1) and following the same steps followed to derive (A6) and (A8), one finds

$$I_3 = -\frac{1}{2(m+1)}\int_0^{\alpha}\tau\psi(\tau)\,\mathrm{d}\tau\int_0^{\infty}\xi b(\xi)J_1(t\xi)J_1(\tau\xi)\,\mathrm{d}\xi. \tag{A9}$$

From eqns (A1), (A6), (A8) and (A9) and by recourse to the definition (63) one can easily derive eqn (73) appearing in the main text.

APPENDIX B

First, we evaluate the following integral by contour integration in the $Z = X+iY$ plane

$$I_1 = \int_0^{\infty}\frac{Z}{\sqrt{1+Z^2}}J_1(\tau Z)J_1(\rho Z)\,\mathrm{d}Z, \quad \tau \geq \rho. \tag{B1}$$

The integrand has branch points at $\pm i$ due to the radical $y(Z) = \sqrt{1+Z^2}$. When the branch cuts and the branches of $y(Z)$ are chosen as shown in Fig. B.1, then by setting

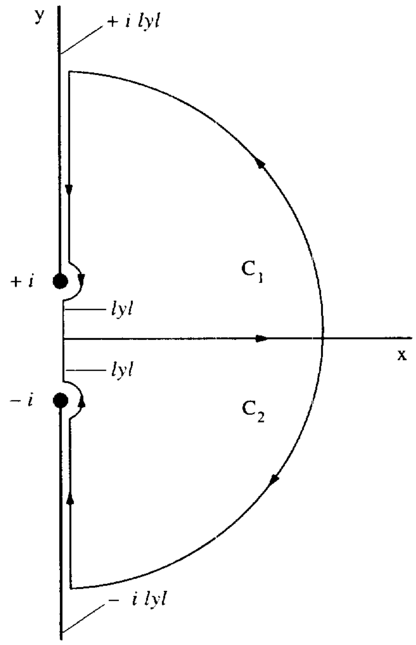

Fig. B1. Contours C_1 and C_2 of integrals I_1 and I_2; —, branch cut.

$$I_1 = \frac{1}{2}[\Lambda_1 + \Lambda_2], \quad \Lambda_j = \int_0^\infty G_1(Z) H_1^{(j)}(\tau Z) J_1(\rho Z)\,\mathrm{d}Z, \quad (j = 1, 2), \quad G_1(Z) = \frac{Z}{\sqrt{1+Z^2}} \tag{B2}$$

with $H_1^{(1)}$, $H_1^{(2)}$ denoting Hankel's functions of the first and second kinds, respectively, and integrating $G_1(Z)H_1^{(1)}(\tau Z)J_1(\rho Z)$ around a contour C_1 in the upper right-hand quadrant passing over the branch point $Y = i$, we get

$$\oint_{C_1} G_1(Z) H_1^{(1)}(\tau Z) J_1(\rho Z)\,\mathrm{d}Z = 0 \tag{B3}$$

because there are no singularities within this contour. Also, the contribution from the semicircular indent at the branch point $+i$ is also zero. Thus

$$i\int_1^0 \frac{iY}{\sqrt{1-Y^2}} H_1^{(1)}(i\tau Y) J_1(i\rho Y)\,\mathrm{d}Y + i\int_\infty^1 \frac{iY}{i\sqrt{Y^2-1}} H_1^{(1)}(i\tau Y) J_1(i\rho Y)\,\mathrm{d}Y$$
$$+\int_0^\infty \frac{X}{\sqrt{1+X^2}} H_1^{(1)}(\tau X) J_1(\rho X)\,\mathrm{d}X = 0. \tag{B4}$$

Similarly, integrating $G_1(Z)H_1^{(2)}(\tau Z)J_1(\rho Z)$ around a contour C_2 in the lower right-hand quadrant we derive

$$(-i)\int_1^0 \frac{-iY}{\sqrt{1-Y^2}} H_1^{(2)}(-i\tau Y) J_1(-i\rho Y)\,\mathrm{d}Y - i\int_\infty^1 \frac{-iY}{-i\sqrt{Y^2-1}} H_1^{(2)}(-i\tau Y) J_1(-i\rho Y)\,\mathrm{d}Y$$
$$+\int_0^\infty \frac{X}{\sqrt{1+X^2}} H_1^{(2)}(\tau X) J_1(\rho X)\,\mathrm{d}X = 0. \tag{B5}$$

Next, by recalling the identity $H_\mu^{(1)}(i\tau Y)J_\mu(i\rho Y) = -H_\mu^{(2)}(-i\tau Y)J_\mu(-i\rho Y)$ and adding eqns (B4) and (B5), we obtain

$$I_1 = \int_0^\infty \frac{X}{\sqrt{1+X^2}} J_1(\tau X) J_1(\rho X)\,\mathrm{d}X = \frac{2}{\pi}\int_1^\infty \frac{Y}{\sqrt{Y^2-1}} I_1(\rho Y) K_1(\tau Y)\,\mathrm{d}Y, \quad \rho \leqslant \tau. \tag{B6}$$

Finally, with the aid of the following transformation of variables

$$Y = \frac{1}{s} \tag{B7}$$

the integral I_1 takes the form

$$I_1 = \frac{2}{\pi}\int_0^1 \frac{1}{s^2\sqrt{1-s^2}} I_1(\rho/s) K_1(\tau/s)\,\mathrm{d}s, \quad \rho \leqslant \tau. \tag{B8}$$

Next, consider the integral

$$I_2 = \int_0^\infty \frac{\sqrt{1+Z^2}}{Z} J_1(\tau Z) J_1(\rho Z)\,\mathrm{d}Z, \quad \tau \geqslant \rho. \tag{B9}$$

The integrand has branch points at $\pm i$ due to the radical $y(Z) = \sqrt{1+Z^2}$ and, furthermore, is unbounded at the origin. To "subtract out" this singularity we write

$$I_2' = \int_0^\infty \left[\frac{\sqrt{1+Z^2}}{Z} - \frac{1}{Z}\right] J_1(\tau Z) J_1(\rho Z)\,\mathrm{d}Z, \quad \tau \geqslant \rho. \tag{B10}$$

Writing I_2' as

$$I_2' = \frac{1}{2}[\Lambda_1 + \Lambda_2], \quad \Lambda_j = \int_0^\infty G_2(Z) H_1^{(j)}(\tau Z) J_1(\rho Z)\,\mathrm{d}Z, \quad (j = 1, 2), G_2(Z) = \left[\frac{\sqrt{1+Z^2}}{Z} - \frac{1}{Z}\right] \tag{B11}$$

then it is valid

$$i\int_1^0 \left\{\frac{\sqrt{1-Y^2}}{iY}-\frac{1}{iY}\right\}H_1^{(1)}(i\tau Y)J_1(i\rho Y)\,\mathrm{d}Y+i\int_\infty^1 \left\{\frac{i\sqrt{Y^2-1}}{iY}-\frac{1}{iY}\right\}H_1^{(1)}(i\tau Y)J_1(i\rho Y)\,\mathrm{d}Y$$

$$+\int_0^\infty \left\{\frac{\sqrt{1+X^2}}{X}-\frac{1}{X}\right\}H_1^{(1)}(\tau X)J_1(\rho X)\,\mathrm{d}X=0 \quad \text{(B12)}$$

since there are no singularities within the contour C_1 in the upper right-hand quadrant passing over the branch point $Y = i$ and, furthermore, the contribution from the semi-circular indent at the branch point $+i$ is zero. Similarly, integrating $G_2(Z)H_1^{(2)}(\tau Z)J_1(\rho Z)$ around a contour C_2 in the lower right-hand quadrant we obtain

$$-i\int_1^0 \left\{\frac{\sqrt{1-Y^2}}{iY}+\frac{1}{iY}\right\}H_1^{(2)}(-i\tau Y)J_1(-i\rho Y)\,\mathrm{d}Y-i\int_\infty^1 \left\{\frac{-i\sqrt{Y^2-1}}{-iY}+\frac{1}{iY}\right\}H_1^{(2)}(-i\tau Y)J_1(-i\rho Y)\,\mathrm{d}Y$$

$$+\int_0^\infty \left\{\frac{\sqrt{1+X^2}}{X}-\frac{1}{X}\right\}H_1^{(2)}(\tau X)J_1(\rho X)\,\mathrm{d}X=0. \quad \text{(B13)}$$

Adding eqns (B12) and (B13), we obtain for $\tau \geqslant \rho$

$$I_2'=\int_0^\infty \left\{\frac{\sqrt{1+X^2}}{X}-\frac{1}{X}\right\}J_1(\tau X)J_1(\rho X)\,\mathrm{d}X=\frac{2}{\pi}\int_0^\infty \frac{\sqrt{Y^2-1}}{Y}I_1(\rho Y)K_1(\tau Y)\,\mathrm{d}Y, \quad \rho\leqslant\tau. \quad \text{(B14)}$$

Finally, by recourse to transformation (B7) and to the following value of the integral

$$\int_0^\infty X^{-1}J_1(\tau X)J_1(\rho X)\,\mathrm{d}X=\frac{1}{2}\frac{\rho}{\tau}, \quad \rho\leqslant\tau \quad \text{(B15)}$$

we obtain

$$I_2=\frac{2}{\pi}\int_0^1 \frac{\sqrt{1-s^2}}{s^2}I_1(\rho/s)K_1(\tau/s)\,\mathrm{d}s+\frac{1}{2}\frac{\rho}{\tau}, \quad \rho\leqslant\tau. \quad \text{(B16)}$$